PATRICK ORSON

INTRODUCTION TO TOPOLOGY

Contents

Introduction

These notes are a work in progress and subject to editing/improvement. I welcome corrections and comments.

Most of the results stated here are given without proof in the notes. During the course, some proofs will be given in the classroom, while other results will be set for the student to prove during homework. This is a topic that rewards a hands-on approach, as you get used to the style of arguments topologists like to use.

Starred sections are deemed beyond the scope of the course, but may be interesting to you.

I hope you enjoy the course!

1 Metric spaces

Metric spaces

METRIC SPACES are a way of abstractly adding a notion of "distance" to a set, capturing what we would intuitively recognise as important properties of "distance" from familiar settings like Euclidean space.

Definition 1.1. Let X be a set. We call a function $d\colon X \times X \to \mathbb{R}$ a *metric* and the pair (X, d) a *metric space* if the following properties are satisfied.

(i) NON-DEGENERACY: $d(x, y) = 0$ if and only if $x = y$.

(ii) SYMMETRY: $d(x, y) = d(y, x)$ for all $x, y \in X$.

(iii) TRIANGLE INEQUALITY: $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.

Exercise 1.1. Let (X, d) be a metric space. Prove that for all points $x, y \in X$, the distance $d(x, y) \geq 0$.

The fact that distances are always positive is often included as an axiom of a metric space. This exercise shows this is redundant as an axiom.

Example 1.2. Here are some fundamental examples of metric spaces. In each case, you should confirm the functions have the three required properties to be a metric.

- Any set X with the function

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

  This is called *discrete metric* on X.

- Euclidean space $\mathbb{R}^n$ with the function $d_2\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by

$$d_2(\mathbf{x}, \mathbf{y}) = \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2}.$$

  This is called the *standard*, or *Euclidean* metric. (The verification that this is a metric is quite easy, except for the triangle inequality, for which you may want to look up the *Cauchy inequality*.)

- The function $d_1 \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by

$$d_1(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} |x_i - y_i|$$

is also a metric. Note this coincides with d_2 when $n = 1$.

This is known as the *Manhattan metric*, because it measures the distance from (a, b) to (c, d) in $\mathbb{R}^2$ as if they lay on a grid: "first walk in the x-direction a distance of $c - a$, then walk in the y-direction a distance of $d - b$.

Exercise 1.2. Let X be the set of cities with a major airport in the USA. Consider the following numbers associated to cities x and y.

(a) The distance, in miles, from x to y as the crow flies.

(b) The distance, in miles, from x to y by road.

(c) The time, in minutes, of the shortest flight from x to y.

(d) The cost, in dollars, of the cheapest flight from x to y.

Discuss informally which conditions (i) to (iv) apply to (a)-(d).

Exercise 1.3. Let $X = \{a, b, c\}$ with a, b and c distinct. Write down functions $d_j \colon X \times X \to \mathbb{R}_{\geq 0}$, such that:

(a) d_1 satisfies conditions (i) and (ii) but not (iii).

(b) d_2 satisfies conditions (ii) and (iii) and $d_2(x, y) = 0$ implies $x = y$, but it is not true that $x = y$ implies $d_2(x, y) = 0$.

(c) d_3 satisfies conditions (ii) and (iii) and $x = y$ implies $d_3(x, y) = 0$. but it is not true that $d_3(x, y) = 0$ implies $x = y$.

(d) d_4 satisfies conditions (i) and (iii) but not (ii).

Normed vector spaces

Most important metrics on vector spaces arise from the slightly more refined notion of a *norm*.

Definition 1.3. Let V be a vector space over $\mathbb{F}$ (with $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$) and $N \colon V \to \mathbb{R}$ a map such that, writing $N(\mathbf{u}) = \|\mathbf{u}\|$, the following properties hold.

(i) POSITIVITY: $\|\mathbf{u}\| \geq 0$ for all $\mathbf{u} \in V$.

(ii) NON-DEGENERACY: If $\|\mathbf{u}\| = 0$, then $\mathbf{u} = \mathbf{0}$.

(iii) SCALAR LINEARITY: If $\lambda \in \mathbb{F}$ and $\mathbf{u} \in V$, then $\|\lambda \mathbf{u}\| = |\lambda| \|\mathbf{u}\|$.

(iv) TRIANGLE INEQUALITY: If $\mathbf{u}$, $\mathbf{v} \in V$, then $\|\mathbf{u}\| + \|\mathbf{v}\| \geq \|\mathbf{u} + \mathbf{v}\|$.

Then we call $\| \ \|$ a *norm* and say that $(V, \| \ \|)$ is a *normed vector space*.

Exercise 1.4. By putting $\lambda = 0$ in Definition 1.3 (iii), show that $\|\mathbf{0}\| = 0$.

Any normed vector space can be made into a metric space in a natural way but not all metrics on vector spaces come from norms.

Lemma 1.4. *If* $(V, \| \|)$ *is a normed vector space, then the condition*

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

defines a metric d *on* V.

Exercise 1.5.

(a) If V is a vector space over $\mathbb{R}$ and d is a metric derived from a norm in the manner described above, then, if $\mathbf{u} \in V$ we have $d(\mathbf{0}, 2\mathbf{u}) = 2d(\mathbf{0}, \mathbf{u})$.

(b) If V is non-trivial (i.e. not zero-dimensional) vector space over $\mathbb{R}$ and d is the discrete metric on V, then d cannot be derived from a norm on V.

Example 1.5. Let $p \in (0, \infty)$. Then for $\mathbf{x} \in \mathbb{R}^n$, the quantities

$$||\mathbf{x}||_p = \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p}$$

and

$$||\mathbf{x}||_\infty = \max\{|x_1|, |x_2|, \dots, |x_n|\}$$

are norms. We write d_p and d_∞ for the corresponding metrics. Note that d_2 agrees with the usual Euclidean metric already defined. (To verify these are norms, the triangle inequality is the most difficult to check. For this, you may want to look up the *Minkowski inequality*.)

Example 1.6. Let S be any set and write $B(S)$ for the set of bounded real-valued functions with domain S. Then the *sup norm* on $B(S)$ is

$$||f|| = \sup_{s \in S} |f(s)|.$$

Definition 1.7. Two metrics d and ρ on a set X are *Lipshitz equivalent* if there exist constants $K > 0$ and $L > 0$ such that for all $x, y \in X$

$$K\rho(x, y) \leq d(x, y) \leq L\rho(x, y).$$

Exercise 1.6. Given a set X, show that Lipshitz equivalence is an equivalence relation on the set of metrics on X.

Theorem 1.8. *For any* $p \in (0, \infty)$ *the metrics* d_p *and* d_∞ *on* $\mathbb{R}^n$ *are Lipshitz equivalent.*

As Lipshitz equivalence is an equivalence relation, this shows d_p and d_q are Lipshitz equivalent for all $p, q \in (0, \infty)$.

In fact, *all* metrics on $\mathbb{R}^n$ derived from norms are Lipshitz equivalent. We leave it to the interested reader to investigate.

Proof. We leave it to the reader to convince themselves that the following inequalities hold

$$||\mathbf{x}||_\infty \leq ||\mathbf{x}||_p \leq n^{1/p} ||\mathbf{x}||_\infty$$

The theorem then follows from the definitions of d_p and d_∞. □

Open sets and continuity in metric spaces

In Analysis, one sees the definition of an open disc of radius $r > 0$ around a point $x \in \mathbb{R}^n$. This is simply the set of points $y \in \mathbb{R}^n$ a distance less than or equal to the centre point x. This notion is easily generalised to metric spaces.

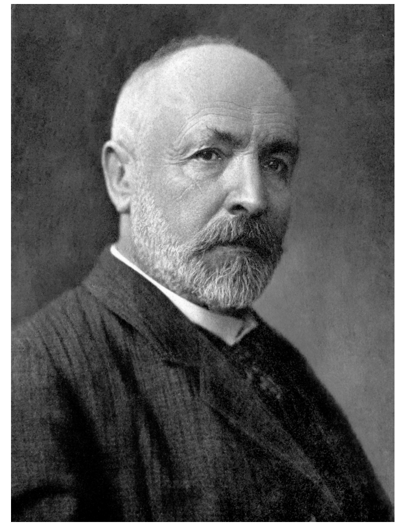


Figure 1.1: Georg Cantor (1845-1918) pioneered set theory and infinite cardinal numbers. His mathematical ideas were controversial throughout his career, but they turned out to be influential and visionary.

Definition 1.9. Let (X, d) be a metric space. Then, given $x \in X$ and $r > 0$, the set

$$B(x;r) = \{y \in X \,|\, d(x,y) < r\}$$

is called the *open ball* around x of radius r.

It was Georg Cantor who first identified the following particularly "well behaved" class of subsets of a metric space.

Definition 1.10. Let (X, d) be a metric space. Then a subset $U \subseteq X$ is *open* if, for every $x \in U$, there exists $\varepsilon > 0$ such that $B(x;\varepsilon) \subseteq U$.

Example 1.11. (i) Let (X, d) be a metric space. Then, for all $x \in X$ and $r > 0$, the open ball $B(x;r)$ is an open set. To see this, let $y \in B(x;r)$. Then $B(y;R) \subseteq B(x;r)$ for $R = r - d(x,y)$.

(ii) Let $\mathbb{R}^n$ have the Euclidean metric. Then for every $\mathbf{x} \in X$, the singleton $\{\mathbf{x}\}$ is not open. However the complement of the singleton $\mathbb{R}^n \setminus \{\mathbf{x}\}$ is open. You should verify these facts.

(iii) Let (X, d) be any set with the discrete metric. Then it is straightforward to confirm that all subsets of X are open.

Lipshitz equivalent metrics are important for the following reason.

Lemma 1.12. *If d and ρ are Lipshitz equivalent metrics on a set X, then $U \subseteq X$ is open with respect to d if and only if it is open with respect to ρ.*

Exercise 1.7. Let (X, d) be a metric space. Define a function $\rho\colon X \times X \to \mathbb{R}$ by $\rho(x,y) = d(x,y)/(1 + d(x,y))$. Show that ρ is a metric. Show that a set is open with respect to d if and only if it is open with respect to ρ. Find an example to show that d and ρ are in general not Lipshitz equivalent.

The next result is a foundational observation, and is the reason Cantor considered open sets so "well-behaved". It's also pretty easy to verify.

Theorem 1.13. *Let (X, d) be a metric space. Then the following are true.*

(i) The empty set $\emptyset$ and the space X are open sets.

(ii) The union of arbitrarily many open sets is open:
If U_α in an open set in X for all $\alpha \in I$ then $\bigcup_{\alpha \in I} U_\alpha$ is open.[1]

[1] I is some arbitrary indexing set, possibly (uncountably) infinite

(iii) The intersection of finitely many open sets is open:
If U_i is an open set in X for all $1 \leq i \leq n$, then $\bigcap_{i=1}^{n} U_i$ is open.

There is a straightforward example to show that condition (iii) really needs the intersection to be finite to be true.

Example 1.14. Working in $\mathbb{R}$ with the usual metric, consider the collection of open intervals $U_i = B(0; 1/i) = (-1/i, 1/i)$, for $i \in \mathbb{Z}^+$. Then $\bigcap_{i=i}^{\infty} U_i = \{0\}$, and this is not open.

We used open balls to define what it generally meant for a subset of a metric space to be open. We can go a little further, and use them to describe open sets in the following way.

Theorem 1.15. *Let (X, d) be a metric space. Then $U \subseteq X$ is open if and only if U is a union of open balls. In other words, there exists a collection of points $x_\alpha \in X$ and radii $\varepsilon_\alpha > 0$ for $\alpha \in I$ some indexing set , such that $U = \bigcup_{\alpha \in I} B(x_\alpha; \varepsilon_\alpha)$.*

In Analysis, we see the ε-δ definition for continuity of a function $f\colon \mathbb{R} \to \mathbb{R}$.[2] This is very easy to generalise to metric spaces.

[2] For all $x \in \mathbb{R}$ and for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

Definition 1.16. Let (X, d) and (Y, ρ) be metric spaces. A function $f\colon X \to Y$ is called *continuous* if, for all $t \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d(s, t) < \delta \implies \rho(f(s), f(t)) < \varepsilon.$$

Lemma 1.17. *If (X, d), (Y, ρ) and (Z, σ) are metric spaces, and $f\colon X \to Y$, $g\colon Y \to Z$ are continuous, then the composition $g \circ f\colon X \to Z$ is continuous.*

In fact, this obvious generalisation of the definition of continuity has a remarkable rephrasing in terms of open sets.

Theorem 1.18. *Let (X, d) and (Y, ρ) be metric spaces. A function $f\colon X \to Y$ is* continuous *if and only if for every open set $U \subseteq Y$, the preimage $f^{-1}(U) \subseteq X$ is open in X.*[3]

[3] Note that this *does not* say that the images of open sets are open when f is continuous. This is generally NOT TRUE.

Exercise 1.8. Reprove Lemma 1.17 using this rephrasing of continuity to make it about open sets. Look how easy it is now!

Closed sets in metric spaces

Cantor identified a second class of very well behaved sets, which we discuss in this section. We need to first discuss *limits* in metric spaces. Again, this idea from Analysis generalise almost immediately to metric spaces.

Definition 1.19. Let $x_0, x_1, x_2, \ldots \in X$ be a sequence of elements in a metric space (X, d). We call $x \in X$ the *limit* of the sequence if, for all $\varepsilon > 0$ there exists $N \geq 0$ such that

$$n \geq N \implies d(x_n, x) < \varepsilon.$$

As usual, we write "$x_n \to x$ as $n \to \infty$" or "$\lim_{n\to\infty} x_n = x$" if this is the case.

Perhaps you have never thought about the possibility that a sequence might have more than one limit. Don't worry, in metric spaces, everything still behaves as one would hope.

Lemma 1.20. *Let (X, d) be a metric space. If a sequence x_n has a limit, then that limit is unique.*

Here is a classic exercise from Analysis, transported into the context of metric spaces.

Exercise 1.9. Show that a function $f\colon X \to Y$ between metric space (X, d) and (Y, ρ) is continuous if and only if for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n) \to f(x)$ in Y.

Definition 1.21. Let (X, d) be a metric space and let $A \subseteq X$ be a subset. We say A is *closed* if, whenever x is the limit of a sequence of elements in A, the element x is also in A.

Closed sets satisfy a similar (but different!) collection of conditions to open sets.

Theorem 1.22. *Let (X, d) be a metric space. Then the following are true.*

(i) The empty set $\emptyset$ and the space X are closed sets.

(ii) The intersection of arbitrarily many closed sets is closed:
If A_α is a closed set in X for all $\alpha \in I$ then $\bigcap_{\alpha\in I} A_\alpha$ is closed.[4]

(iii) The union of finitely many closed sets is closed:
If A_i is a closed set in X for all $1 \leq i \leq n$, then $\bigcup_{i=1}^{n} A_i$ is closed.

[4] I is some arbitrary indexing set, possibly (uncountably) infinite

Compare conditions (i) of Theorems 1.13 and 1.22. We immediately see that some sets are open *and* closed! This differs from the normal English usage, where you might expect "not open" to be equivalent to "closed". For a given set $U \subseteq X$, you should generally expect openness for A and closedness for A to be unrelated features, to be analysed separately.

A set that is open and closed is sometimes called "*clopen*", but I think this word is kind of yucky...

On the other hand, there is a deep connection between the general concepts of open and closed sets.

Theorem 1.23. *Let (X, d) be a metric space. Then $U \subseteq X$ is open if and only if $X \setminus U$ is closed.*

Exercise 1.10. Using De-Morgan's laws and Theorem 1.23, give a quick proof of Theorem 1.22, by considering Theorem 1.13.

Exercise 1.11. Prove that a function $f\colon X \to Y$ between metric space (X, d) and (Y, ρ) is continuous if and only if $f^{-1}(A)$ is closed in X, for every closed set $A \subseteq Y$.

Example 1.24. A nice argument to show that the n-sphere

$$S^n = \{x \in \mathbb{R}^{n+1} \mid ||x|| = 1\} \subseteq \mathbb{R}^{n+1}$$

is closed if to observe that the norm map

$$\mathbb{R}^{n+1} \to \mathbb{R}$$

We include the following remark, for the record.

Remark 1.25. There is some confusing terminology in this part of metric space theory. Let $A \subseteq X$ be a subset of a metric space.

- A *limit point* of A is a point $x \in X$ such that every open ball around x contains infinitely many points of A.
- An *isolated point* of A is a point $x \in A$ such that $B(x; \varepsilon) \cap A = \{x\}$ for some $\varepsilon > 0$.

Clearly an isolated point $x \in A$ is not a limit point of A. However, confusion might arise, because an isolated point x is a *limit of a sequence* in A (the constant sequence $x, x, x, x, \dots$). Just be careful...

2 *Topological spaces*

Topological spaces

WE NOW investigate how much of the work we have done on metric spaces can be recovered when we strip away the metric but retain a concept of open set. The idea is to change perspective, by making Theorem 1.13 a definition, rather than a consequence of other definitions.

Definition 2.1. Let X be a set and τ be a collection of subsets of X. We call τ a *topology* on X, and the pair (X, τ) a *topological space* if the following conditions are satisfied.

(i) The empty set $\emptyset \in \tau$ and the space $X \in \tau$.

(ii) If $U_\alpha \in \tau$ for all $\alpha \in I$ then $\bigcup_{\alpha \in I} U_\alpha \in \tau$.[1]

(iii) If $U_i \in \tau$ for all $1 \leq i \leq n$, then $\bigcap_{i=1}^{n} U_i \in \tau$.

[1] I is some arbitrary indexing set, possibly (uncountably) infinite

We refer to the elements $U \in \tau$ of a topology as *open sets* in that topology.

Example 2.2. Given a metric space (X, d), let τ be the collection of all open sets in the sense of a metric space (Definition 1.10). Then τ is a topology by Theorem 1.13. We call this the topology on X *induced by the metric*. If a topology is induced by some metric, we call the topological space *metrisable*.

Example 2.3. As open sets in the metrics d_p all coincide, this shows these all induce the same topology on $\mathbb{R}^n$.

Here are two "trivial" topologies you can put on any set X.

Definition 2.4. Let X be a set.

(a) The topology $\tau = \mathscr{P}(X)$ consisting of all possible subsets of X is called the *discrete topology*.

(b) The topology $\tau = \{\emptyset, X\}$ is called the *indiscrete topology*.

Recall, $\mathscr{P}(X)$ denotes the *power set*; the set of all possible subsets of X.

Exercise 2.1. Let X be a set. Confirm that the discrete topology and the indiscrete topology are indeed topologies.

Exercise 2.2. Let X be a set.

(i) Prove that the discrete metric on X induces the discrete topology.

(ii) Prove that if X has at least 2 elements then no metric on X induces the indiscrete topology.

So long as you are careful to satisfy the three properties of a topology, you can cook up all sorts of topologies on sets.

Example 2.5. Let $X = \{1,2,3,4\}$. Then $\tau = \{\emptyset, X, \{1\}, \{3\}, \{1,3\}\}$ is a topology.

Exercise 2.3. Let $X = \{1,2,3\}$. How many elements are in $\mathscr{P}(X)$? How many elements are in $\mathscr{P}(\mathscr{P}(X))$? How many topologies are there on X?

We base the more abstract definition of a closed set on Theorem 1.23.

Definition 2.6. Let (X,τ) be a topological space. Then a set $A \subseteq X$ is *closed* if $X \setminus A$ is open.

The reader should immediately verify the following theorem.

Theorem 2.7. *Let (X,τ) be a topological space. Then the following are true.*

(i) The empty set $\emptyset$ and the set X are closed.

(ii) If $A_\alpha \subseteq X$ is closed for all $\alpha \in I$ then $\bigcap_{\alpha \in I} A_\alpha$ is closed.[2]

(iii) If $A_i \subseteq X$ is closed for all $1 \le i \le n$, then $\bigcup_{i=1}^{n} A_i$ is closed in X.

[2] I is some arbitrary indexing set, possibly (uncountably) infinite

Remark 2.8. The entire theory of point-set topology could be developed using closed sets in our preferred collection of "interesting sets" and making the conditions in Theorem 2.7 axioms rather than consequences. There is no particularly good reason topologies are defined with open sets and not closed – that's just how things went!

Interior and closure

Given an arbitrary subset $A \subseteq X$ of a topological space X, we wish to consider the "largest" open set contained in A and the "smallest" closed set containing A. This idea of "largest" and "smallest" is captured in the following definition.

Definition 2.9. Let (X,τ) be a topological space and $A \subseteq X$ a subset.

(a) The *interior* of A is the union of all open sets contained in A.

(b) The *closure* of A is the intersection of all closed sets containing A.

The interior is open, as an arbitrary union of open sets is open. Similarly, the closure is closed, as it is an intersection of closed sets.

Exercise 2.4. Let (X, τ) be a topological space and let $A \subseteq X$ be a subset.

(i) Show that $(\text{int}(A))^c = \text{cl}(A^c)$

(ii) Show that $\text{int}(A^c) = (\text{cl}(A))^c$

Here are some popular alternative ways to think about interior and closure.

Definition 2.10. Let (X, τ) be a topological space and let $A \subseteq X$ be a subset. We call a point $x \in A$ an *interior point* of A if there exists $U \in \tau$ with $x \in U \subseteq A$.

Lemma 2.11. *Let (X, τ) be a topological space and let $A \subseteq X$ be a subset. Then the interior of A is the set of all its interior points. In other words,*

$$\text{int}(A) = \{x \in A \mid \exists\, U \in \tau \textit{ such that } x \in U \subseteq A\}.$$

Here is a similar recasting of closure in terms of a property satisfied by all its elements.

Definition 2.12. Let (X, τ) be a topological space and $A \subseteq X$ a subset. We say $x \in X$ is *adherent to* A if every open set $U \subseteq X$ with $x \in X$ has $U \cap A \neq \emptyset$.

Lemma 2.13. *Let (X, τ) be a topological space and let $A \subseteq X$ be a subset. Then the closure of A is the set*

$$\text{cl}(A) = \{x \in X \mid \forall U \in \tau \textit{ with } x \in U, \textit{we have } A \cap U \neq \emptyset\}.$$

In other words, the closure of A is the set of points adherent to A.

Definition 2.14. Let (X, τ) be a topological space and $A \subseteq X$ a subset. The *frontier* of A is the set $\text{cl}(A) \setminus \text{int}(A)$.

The frontier of A is often called the "boundary" of A, but frontier is a better word because "boundary" is a terminology clash with another popular use of that word in topology.

Remark 2.15. The properties of the frontier of a set depend highly on the set. For example, the frontier may be disjoint from A, contained in A, or neither. The frontier is always a closed set, as can be seen by restating

$$\text{cl}(A) \setminus \text{int}(A) = \text{cl}(A) \cap \text{int}(A)^c,$$

the intersection of two closed sets.

Exercise 2.5. The frontier of A is the set of points adherent to both A and A^c.

There is some potentially confusing terminology to introduce now.

Definition 2.16. Let $A \subseteq X$ be a subset of a topological space.

- A *limit point* of A is a point $x \in X$ such that every open set around x contains a point of A other than itself.
- An *isolated point* of A is a point $x \in A$ such that there exists an open set $U \subseteq X$ with $U \cap A = \{x\}$. In other words, $\{x\} \subseteq A$ is an open set in the subspace topology on A.

Exercise 2.6. Show that the set of limit points of $A \subseteq X$ is closed. Show that the closure of $A \subseteq X$ is the union of the set of limit points of A and isolated points of A.

As in Remark 1.25, the confusion could arise because an isolated point $x \in A$ is not a limit point of A, but it is always the *limit of a sequence* in A (namely, the constant sequence $x, x, x, \dots$).

Continuous functions

Similarly to how we based the definition of a topology on Theorem 1.13, we will base the definition of a continuous function between topological spaces on Theorem 1.18.

Definition 2.17. Let (X, τ) and (Y, σ) be topological spaces. Then a function $f\colon X \to Y$ is *continuous* if the preimage $f^{-1}(U) \subseteq X$ is open in X, whenever $U \subseteq Y$ is open in Y.

This definition makes the proof of the following a very easy exercise.

Theorem 2.18. *Let (X, τ), (Y, σ), (Z, μ) be topological spaces, and $f\colon X \to Y$, $g\colon Y \to Z$ be continuous maps. Then the composite $g \circ f$ is continuous.*

Continuity can also be phrased in terms of closed sets.

Proposition 2.19. *A function $f\colon X \to Y$ is continuous if and only if the preimage of every closed set is closed.*

Example 2.20. Here are some trivial examples of continuous functions.

(i) A map $f\colon X \to Y$ is called *constant* if it sends all of X to some point $y \in Y$. Constant maps are continuous: for all $U \subseteq Y$ the preimage $f^{-1}(U)$ is either empty (if $y \notin U$) or the entire set X (if $y \in U$). In either case the preimage is open.

(ii) Any function $X \to Y$ where X has the discrete topology is continuous, because the preimage of any set in Y is open (as all subsets of X are open).

(iii) Any function $X \to Y$ where Y has the indiscrete topology is continuous, because $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(Y) = X$ are both open, no matter what topology is on X.

Here is when we consider two topological spaces to be "the same".

Definition 2.21. We call topological spaces (X, τ) and (Y, σ) *homeomorphic* if there exists a bijection $f\colon X \to Y$ such that both f and

f^{-1} are continuous. We write $X \cong Y$ to indicate that there exists a homeomorphism.

Remark 2.22. A property of a space is *topological* if it is preserved under homeomorphism. Homeomorphisms can do fairly brutal things to a metric, meaning topological properties must be quite robust. For example, you may recall from analysis the notion of *compactness* of a subset of $A \subseteq \mathbb{R}$ (every sequence in A has a convergent subsequence, whose limit is in A); it turns out compactness is a topological property of a space, in the sense that all spaces homeomorphic to A will also be compact. But, for example, *length* is not a topological property, as the next example shows.

Example 2.23. There is a homeomorphism $(-1,1) \cong \mathbb{R}$ given by

$$f\colon (-1,1) \to \mathbb{R}; \qquad f(x) = \frac{x}{1-|x|}.$$

This is easily checked to be a continuous bijection using methods from Analysis. The inverse function is $f^{-1}(x) = \frac{x}{1+|x|}$, which is also continuous, verified similarly.

Example 2.24. Generalising the previous example, let $B(x_0; r) \subseteq \mathbb{R}^n$ be any open ball. Then there is a homeomorphism $B(x;r) \cong \mathbb{R}^n$, given by translating the ball to the origin, projecting the point x onto the unit sphere, then scaling the radius by a homeomorphism between $[0,1)$ and $[0,\infty)$:

$$x \mapsto x - x_0 \mapsto \frac{x-x_0}{||x-x_0||} \mapsto \frac{x-x_0}{||x-x_0||} \cdot \frac{||x-x_0||}{r-||x-x_0||}.$$

Thus we get

$$f\colon B(x;r) \to \mathbb{R}^n; \qquad f(x) = \frac{x-x_0}{r-||x-x_0||}.$$

The inverse function is $f^{-1}(x) = \frac{rx}{1+||x||} + x_0$, which is also continuous.

It is very tempting to think "continuous bijection implies homeomorphism". This is not true; it is possible to have a continuous bijective map $f\colon X \to Y$ such that the inverse f^{-1} is not continuous. Here is a straightforward example.

Example 2.25. Let $X = \{1,2\}$. The discrete topology is $\tau = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ and the indiscrete topology $\sigma = \{\emptyset, \{1,2\}\}$. Then the identity map $\mathrm{Id}_X\colon (X,\tau) \to (X,\sigma)$ is continuous. But the inverse of this map (i.e. the identity map) is not continuous $\mathrm{Id}_X\colon (X,\sigma) \to (X,\tau)$. For example, the preimage of $\{1\}$ is $\{1\}$, which is not open in σ.

Definition 2.26. A function $(X, \tau) \to (Y, \sigma)$ is called *open* if the image of every open set is an open set. Similarly, f is called *closed* if the image of every closed set is closed.

The following is essentially tautological, but you should go through the motions of verifying it.

Proposition 2.27. *A continuous bijective map is a homeomorphism if and only if it is open.*

Basis for a topology

We saw in metric spaces that the concept of open balls was important for actually writing down the topology. In Theorem 1.15, it was shown that one could in fact define open sets in metric spaces to be those sets expressible as a (possibly infinite) union of open balls. This idea generalises to the concept of a *basis* for a topology.

The reader should compare the ideas "basis of a topology" to "basis of a vector space". They are not quite the same, but are united by the idea of taking a subcollection of all the objects and using them to generate the full collection of objects. They are also related by the fact that bases are not required for the definition of vector spaces nor for the definition of topological spaces – they are an artificial choice one makes in order to make computation easier.

Definition 2.28. Let (X, τ) be a topological space. A collection $\mathscr{B} \subseteq \tau$ is a *basis* for τ if every open set $U \in \tau$ is a union of elements of $\mathscr{B}$.

Exercise 2.7. Let X and Y be topological spaces and let $\mathscr{B}$ be a basis for the topology on Y. Prove that a function $f \colon X \to Y$ is continuous if and only if $f^{-1}(U)$ is open for all $U \in \mathscr{B}$.

Here is a theorem for recognising when a collection $\mathscr{C}$ of open sets is a basis for the topology.

Theorem 2.29. *Let X be a topological space and let $\mathscr{C}$ be a collection of open sets in X. Suppose that for each open set $U \subseteq X$ and each $x \in U$, there is an element $V \in \mathscr{C}$ such that $x \in V \subseteq U$. Then $\mathscr{C}$ is a basis for the topology.*

Example 2.30. In a metric space (X, d), we defined a set $U \subseteq X$ to be open if for every $x \in U$, there is an open ball $x \in B(x; \varepsilon) \subseteq U$. So the induced topology on a metric space is essentially defined to be the one that has the collection set of open balls as its basis.

The following is an application of the basis recognition theorem.

Proposition 2.31. *If $\mathscr{B}$ is a basis for (X, τ), then a collection of subsets $\mathscr{C}$ is a basis for (X, τ) if for all $U \in \mathscr{B}$, and all $x \in U$, there exists $V \in \mathscr{C}$ with $x \in V \subseteq U$.*

If I have one basis and I think I might have a second one, I can use Proposition 2.31 to prove it.

Example 2.32. Instead of using open balls as a basis for $\mathbb{R}^3$, we could use Euclidean cubes, polar cubes, cylindrical cubes or spherical cubes as the basis. By Proposition 2.31, to prove these are bases, the task is to show that at every point $x \in \mathbb{R}^3$, one of these objects fits inside an arbitrarily small ball $B(x; \varepsilon)$.

For example, consider spherical spherical co-ordinates (ρ, θ, φ). Let $\mathscr{C}$ be the collection of subsets of $\mathbb{R}^3$ consisting of open cubes

$$\{(\rho, \theta, \varphi) \in (a, b) \times (c, d) \times (e, f)\}.$$

At any point $x \in \mathbb{R}^3$, by taking the cube small enough, we can find a cube containing x, and inside any given open ball $B(x; \varepsilon)$. So by Proposition 2.31, the open cubes form a basis.

Fineness and coarseness

The discrete and indiscrete topologies on a set X (Definition 2.4) in some sense "opposite". The discrete topology has the most number of open sets a topology can have and the indiscrete topology has the least. Let's make this formal by introducing a way of comparing "size" of a topology.

Definition 2.33. Let X be a set and let τ_1 and τ_2 be two topologies on X. If $\tau_1 \subseteq \tau_2$ then we say τ_1 is *coarser* (or *smaller*) than τ_2. Equivalently, we say τ_2 is *finer* (or *larger*) than τ_1.

Remark 2.34. Fix a set X. Given two topologies τ_1 and τ_2, the relation "τ_1 is finer than τ_2" is a partial order on the set of topologies on X.

Next we prove some simple, but useful lemmas for constructing new topologies with desired properties.

Lemma 2.35. *Let X be a set and let $\mathscr{A}$ be a collection of subsets of X. Then there exists a unique topology $\tau_{\mathscr{A}}$ such that*

(i) $\tau_{\mathscr{A}} \supseteq \mathscr{A}$, *and*

(ii) if τ is a topology with $\tau \supseteq \mathscr{A}$, then $\tau \supseteq \tau_{\mathscr{A}}$.

Definition 2.36. Let X be a set and let $\mathscr{A}$ be a collection of subsets of X. We call the topology $\tau_{\mathscr{A}}$ described in Lemma 2.35 the *coarsest (or smallest) topology containing $\mathscr{A}$.*

Lemma 2.37. *Let X be a set. Let I be a non-empty indexing set and (X_α, τ_α) be a topological space for all $\alpha \in I$. Given functions $f_\alpha \colon X \to X_\alpha$, for every $\alpha \in I$, there is a coarsest topology on X such that the maps f_α are all continuous.*

3 Subspaces, products and quotients

We now present several constructions for making new spaces from old. In each case we give definitions in terms of "the coarsest topology that makes [some function] continuous". Such definitions are good for proving abstract properties of a topology, but the reader should also learn the subsequent equivalent formulations, which give intuition about which sets are open, because these are much more useful in practical computations.

Subspace topology

When we have a subset $Y \subseteq X$, there is an associated *inclusion function*

$$\iota_Y \colon Y \to X; \qquad \iota_Y(x) = x.$$

Using Lemma 2.37, we define a topology on Y.

Definition 3.1. Let (X, τ) be a topological space and $Y \subseteq X$ be a subset. The *subspace topology* τ_Y for Y is the coarsest (or smallest) topology for Y such that the inclusion map $Y \to X$ is continuous.

A subset $Y \subseteq X$ with the subspace topology is, of course, called a *subspace*.

Proposition 3.2. *Let (X, τ) be a topological space and $Y \subseteq X$ be a subset. Then the subspace topology τ_Y is the collection of sets $Y \cap U$ such that $U \in \tau$.*

Exercise 3.1.

(i) Let (X, τ) be a topological space and $Y \subseteq X$ be an open set in X, show that the subspace topology τ_Y is the collection of sets $U \in \tau$ such that $U \subseteq Y$.

(ii) Let $X = \mathbb{R}$ and $Y = [0, 1]$. show that $[0, \frac{1}{2}) \in \tau_Y$ but $[0, \frac{1}{2}) \notin \tau$.

Proposition 3.3. *Let X and Y be topological spaces and let $f \colon X \to Y$ be continuous.*

(i) The restriction $f|_A \colon A \to Y$ is continuous.

(ii) If $B \subseteq Y$ is a subspace and $\operatorname{im}(f) \subseteq B$, then the map $g\colon X \to B$ obtained from f by restricting the codomain is continuous.

(iii) Let Z be a topological space such that $Y \subseteq Z$ is a subspace. Then the function $h\colon X \to Z$ defined by $h(x) = f(x)$ is continuous.

The following lemma is VERY helpful.

Lemma 3.4. *If (X, τ) has a basis $\mathcal{B}$ then a basis for the subspace $A \subseteq X$ is given by*

$$\{U \cap A \mid U \in \mathcal{B}\}.$$

Using this lemma, we get basis for some familiar spaces.

Example 3.5. A basis for Euclidean space is given by the collection $\mathcal{B}$ of all open balls $B(x; r)$. This leads to:

(i) The closed interval $[0, 1]$ has a basis given by sets $(a, b) \subseteq (0, 1)$, $[0, a) \subseteq [0, 1]$ and $(a, 1] \subseteq [0, 1]$.

(ii) The circle $S^1 \subseteq \mathbb{R}^2$ has a basis given by open arcs

$$A_{(a,b)} = \{(\cos\theta, \sin\theta) \mid \theta \in (a, b)\}$$

where $(a, b) \subset \mathbb{R}$ is any open interval.

Product topology for finite products

Given sets X and Y, the Cartesian product $X \times Y$ is the set of ordered pairs

$$X \times Y = \{(x, y) \mid x \in X,\ y \in Y\}.$$

Definition 3.6. Given topological spaces X and Y, an *open box* is a subset $U \times V \subseteq X \times Y$, where $U \subset X$ and $V \subseteq Y$ are open sets.

It makes intuitive sense that at least the open boxes should be open in $X \times Y$. We now define a topology $X \times Y$, essentially designed to make the set of open boxes a basis.

Definition 3.7. Given topological spaces (X, τ) and (Y, σ), the *box topology* μ_{box} is the collection of subsets $Z \subseteq X \times Y$ such that for every $(x, y) \in Z$, there exists an open box $U \times V$ such that

$$(x, y) \in U \times V \subseteq X \times Y.$$

Proposition 3.8. *Let (X, τ) and (Y, σ) be topological spaces. The box topology μ_{box} is indeed a topology on $X \times Y$. Moreover, the collection of* open boxes

$$\mathscr{B} = \{U \times V \mid U \in \tau \text{ and } V \in \sigma\}$$

is a basis for the box topology.

Remark 3.9. As the open boxes are a basis for the box topology, a subset $W \subseteq X \times Y$ is open in the box topology if and only if it is a (possibly infinite) union of open boxes.

There is a second intuitively sensible way to topologise the product $X \times Y$ of topological spaces. The product has two *projection* maps, back to the respective factors:

$$\begin{aligned} \mathrm{pr}_1 \colon X \times Y \to X; \qquad \mathrm{pr}_1(x,y) &= x, \\ \mathrm{pr}_2 \colon X \times Y \to Y; \qquad \mathrm{pr}_2(x,y) &= y \end{aligned}$$

The topology on $X \times Y$ should have the property that both of these are continuous maps. Let us define a topology precisely to ensure this – this is achieved using Lemma 2.37.

Definition 3.10. Given topological spaces (X, τ) and (Y, σ), the *product topology* is the smallest topology on $X \times Y$ such that pr_1 and pr_2 are both continuous.

It is handy that the box topology and the product topology agree on a product $X \times Y$, so either perspective can be used.

Proposition 3.11. *Let (X, τ) and (Y, σ) be topological spaces. Then the box and product topologies on $X \times Y$ are equal.*

The box topology is much easier to visualise, so why are we not using it as the definition of the product topology? First, the boxes are only a *basis* for the product topology, which is an ugly way to define a topology. Secondly, the natural generalisations of the box and product topologies to infinite Cartesian products *do not agree*. In such situations, we should revert to the definition that produces the nicest theorems. It turns out in that the product topology is much more nicely behaved for infinite Cartesian products, which is why this ends up being the "morally correct" definition overall.

Once you know the definition of the topology on the product of two sets, it generalises immediately to finitely many sets. Let (X_i, τ_i) for $i = 1, 2, \ldots, n$ be a collection of topological spaces. The Cartesian product is the set of ordered n-tuples

$$\prod_{i=1}^{n} X_i = X_1 \times \cdots \times X_n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in X_i \text{ for } i = 1, \ldots, n\}.$$

This has n *projection* maps, back to the respective factors

$$\mathrm{pr}_i \colon \prod_{i=1}^{n} X_i \to X_i; \quad \mathrm{pr}_i(x_1, x_2, \ldots, x_n) = x_i.$$

Using Lemma 2.37, we define a topology on $\prod_{i=1}^{n} X_i$.

Definition 3.12. The *product topology* $\prod_{i=1}^{n} \tau_i$ on $\prod_{i=1}^{n} X_i$ is the coarsest (or smallest) topology such the projection map pr_i is continuous for every $i = 1, \ldots, n$.

Exercise 3.2. Show that on $X_1 \times X_2 \times X_3$, the product topology $\tau_1 \times \tau_2 \times \tau_3$ agrees with the topology $(\tau_1 \times \tau_2) \times \tau_3$. In other words, two iterations of Definition 3.10 agrees with Definition 3.12.
Now show by induction that n iterations of Definition 3.10 agrees with Definition 3.12.

Exercise 3.3. Show that the collection of *open boxes*

$$\mathscr{B} = \{U_1 \times \cdots \times U_n \mid U_i \in \tau_i \text{ for } i = 1, 2, \ldots, n\}$$

is a basis for a topology on $\prod_{i=1}^n X_i$ and that the resulting topology agrees with the product topology.

*Product topology for infinite products**

The notion of pairs, triples, ... that are often used for finite Cartesian products does not generalise well to the arbitrarily indexed products, especially when the indexing set is uncountable. Here is a good way to generalise the notion of product.

Definition 3.13. Let I be an index set and for each $\alpha \in I$, let X_α be a set. The *Cartesian product* over I is

$$\prod_{\alpha \in I} X_\alpha = \left\{ x \colon I \to \bigcup_{\alpha \in I} X_\alpha \;\middle|\; x(\alpha) \in X_\alpha \text{ for all } \alpha \in I \right\}$$

When $I = \{1, 2, \ldots, n\}$, the ordered n-tuple $(x_1, x_2, \ldots, x_n)$ corresponds to the function $x(i) = x_i$.

Remark 3.14. The "ordered tuple" perspective on products only works when we can list the elements of I in increasing order $I = \{\alpha_1, \alpha_2, \ldots\}$. So the ordered tuple perspective works if and only if I is countable, and then an ordering must be specified in order to fix an "ordered tuple" perspective on the product.

Exercise 3.4. By writing down a bijection, check that Definition 3.13 agrees with the familiar Cartesian product when $I = \{1, 2, \ldots, n\}$, and with the usual ordering on I.

Exercise 3.5. Define an *open box* in the product $\prod_{\alpha \in I} X_\alpha$ to be a subset $\prod_{\alpha \in I} U_\alpha$, where $U_\alpha \subseteq X_\alpha$ is open for every $\alpha \in I$. Show that the collection $\mathscr{B}$ of all open boxes is a basis for a topology on $\prod_{\alpha \in I} X_\alpha$. Find an example that shows in general the box topology is finer than the product topology.

Quotient topology

We recall the definitions and terminology associated with equivalence relations on a set.

Definition 3.15. A relation $\sim$ on a set X is called an *equivalence relation* if it has the following properties.

(i) REFLEXIVE: $\forall x \in X, \quad x \sim x$,

(ii) SYMMETRIC: $\forall x, y \in X, \quad x \sim y \implies y \sim x$,

(iii) TRANSITIVE: $\forall x, y, z \in X, \quad (x \sim y \text{ and } y \sim z) \implies x \sim z$.

Given an equivalence relation $\sim$ on X, we have:

- An *equivalence class* $[x] = \{y \in X \mid x \sim y\} \subseteq X$, for every $x \in X$.
- The *equivalence set* $X/\sim = \{ [x] \mid x \in X\}$.
- The *quotient map* $q\colon X \to X/\sim;\ x \mapsto [x]$.

Recall, the equivalence set is a partition of X.

Definition 3.16. Let (X, τ) be a topological space with an equivalence relation $\sim$. Then the *quotient topology* is

$$\sigma = \{U \subseteq X/\sim \mid q^{-1}(U) \in \tau\}.$$

We call a quotient set equipped with the quotient topology a *quotient space*.

Proposition 3.17. *The quotient topology is indeed a topology. The quotient topology is the finest (or largest) topology such that $q\colon X \to X/\sim$ is continuous.*

One way to try and understand a quotient space $X/\sim$ is to recognise it as homeomorphic to a known space Y, so we will spend some time thinking about this task now.

Example 3.18. On $\mathbb{R}$, consider the equivalence relation

$$x \sim y \iff x - y \in \mathbb{Z}.$$

What is a likely looking homeomorphic space? Observe that there is a disjoint union

$$\mathbb{R} = \cdots \cup [-1, 0) \cup [0, 1) \cup [1, 2) \cup \ldots$$

and that for very $x \in \mathbb{R}$ there exists a unique $\overline{x} \in [0, 1)$ such that $x \sim \overline{x}$. The map

$$f\colon \mathbb{R}/\sim \to [0, 1); \qquad [x] \mapsto \overline{x}$$

is then a bijection. However, the function f is not continuous, so this likely looking approach does not work. To justify this to yourself, consider that if it were continuous then, as the quotient map is continuous, the composition $f \circ q$ would be continuous (this map is $x \mapsto \overline{x}$). But the interval $[0, \frac{1}{2}) \subseteq [0, 1)$ is open in the subspace topology, and

$$(f \circ q)^{-1}([0, \tfrac{1}{2})) = \bigcup_{n \in \mathbb{Z}} [n, n + \tfrac{1}{2})$$

is not open in $\mathbb{R}$. So $f \circ q$ is not continuous.

The problem in the previous approach was very specific to the closed end of the interval $[0, 1)$ not wrapping around to the open end. To fix this, let's "glue" the ends of the interval together to make a circle

$$S^1 := \{(\cos(2\pi\theta), \sin(2\pi\theta)) \in \mathbb{R}^2 \mid \theta \in \mathbb{R}\},$$

which is given the subspace topology from usual metric-induced topology on $\mathbb{R}^2$. We wish to show that the function

$$\mathbb{R}/\sim \to S^1; \qquad [x] \mapsto (\cos(2\pi x), \sin(2\pi x))$$

is a homeomorphism. It will be helpful to introduce some more terminology.

Definition 3.19. Let X be a set with an equivalence relation $\sim$. A function $f\colon X \to Y$ determines a well-defined function

$$X/\sim \to Y; \qquad [x] \mapsto f(x)$$

if and only if $f(a) = f(b)$ whenever $a \sim b$. In this case we say the function f *descends* to $X/\sim$ and that the map $[x] \mapsto f(x)$ is *induced* by f.

Exercise 3.6. If $f\colon X \to Y$ descends to the quotient space, there is an induced map $X/\sim \to Y$. Confirm that the following facts hold.

- The induced map is surjective if and only if f is surjective.
- The induced map is injective if and only if $a \sim b$ whenever $f(a) = f(b)$.
- The induced map is continuous if and only if f is continuous.
- If f is open then the induced map is open.

Example 3.20. The function

$$f\colon \mathbb{R} \to S^1; \qquad x \mapsto (\cos(2\pi x), \sin(2\pi x))$$

descends to the quotient

$$\mathbb{R}/\sim \to S^1; \qquad [x] \mapsto (\cos(2\pi x), \sin(2\pi x))$$

because whenever $a \sim b$, we have $a = b + n$ for some $n \in \mathbb{Z}$, so that

$$\begin{aligned}(\cos(2\pi a), \sin(2\pi a)) &= (\cos(2\pi(b+n)), \sin(2\pi(b+n))) \\ &= (\cos(2\pi b), \sin(2\pi b)).\end{aligned}$$

Now observe that f is surjective, that $f(a) = f(b)$ implies $a \sim b$ and that f is continuous. Thus the induced map on the quotient is a continuous bijection. It remains to show it is an open map. For this it is sufficient to show that f itself is open. For this it is sufficient to show that the image $f(I)$ is open, for any open interval $I = (a, b) \subseteq \mathbb{R}$. But the image of this is an open arc of S^1. Recall the topology on S^1 consists of sets $U \cap S^1$ where $U \subseteq \mathbb{R}^2$ is open in the standard topology. It is always possible to find an open ball in $\mathbb{R}^2$ intersecting any given circle in a given arc. Even easier: take the open half-plane through the endpoints of the open arc and with the arc contained in the half-plane. We see open arcs are indeed open in S^1 and hence the induced map is open. Thus is it a homeomorphism.

Example 3.21. Let $\sim$ be the equivalence relation on $\mathbb{R}^2$ so that $(x,y) \sim (u,v)$ if and only if $x^2 + y^2 = u^2 + v^2$. Note that the equivalence classes consist of circles centred at the origin. As we can choose a unique representative for each class on the positive x-axis (namely, the place the circle intersects the x-axis), we guess the quotient space $\mathbb{R}^2/\sim$ is homeomorphic to the positive x-axis $[0,\infty)$, with its usual topology.

To show this is true, consider the map

$$f\colon \mathbb{R}^2 \to [0,\infty); \qquad f(x,y) = \sqrt{x^2+y^2}.$$

This descends to the quotient $\mathbb{R}^2/\sim$ because if $(x,y) \sim (u,v)$ then $\sqrt{x^2+y^2} = \sqrt{u^2+v^2}$. The induced map is surjective, because f is surjective, and the induced map is injective because $f^{-1}(\{f(x,y)\}) = f^{-1}(\{\sqrt{x^2+y^2}\}) = [(x,y)]$ for all $(x,y) \in \mathbb{R}^2$. As f is continuous, so is the induced map on the quotient. It remains to show that the induced map is open. In other words, we have to show that whenever $U \subseteq \mathbb{R}^2/\sim$ is open, the image in $[0,\infty)$ is open. But $U \subseteq \mathbb{R}^2/\sim$ is open if and only if $q^{-1}(U) \subseteq \mathbb{R}^2$ is open. Therefore it suffices to show the stronger fact, that f itself is open.

To show f is open, it is sufficient to verify that the image of an open ball in $\mathbb{R}^2$ is open in $[0,\infty)$. Let $B(\mathbf{x};\varepsilon) \subseteq \mathbb{R}^2$ be an arbitrary open ball. Then the reader may verify that

$$f(B(\mathbf{x};\varepsilon)) = \begin{cases} (||\mathbf{x}|| - \varepsilon, ||\mathbf{x}|| + \varepsilon) & \text{if } 0 \notin B(\mathbf{x};\varepsilon), \\ [0, ||\mathbf{x}|| + \varepsilon) & \text{if } 0 \in B(\mathbf{x};\varepsilon), \end{cases}$$

which in each case is an open set in $[0,\infty)$. Thus $\mathbb{R}^2/\sim \;\cong [0,\infty)$ as claimed.

A particularly important class of quotient spaces is given by "crushing" a subspace $A \subseteq X$ to a single point within X. One of the purposes of the quotient space technology is to make this kind of vague intuitive statement precise.

Definition 3.22 (Crushing a subspace)**.** Let (X,τ) be a topological space and let $A \subseteq X$ be a non-empty subset. Let $\sim$ be the equivalence relation on X given by

$$x \sim y \quad \Longleftrightarrow \quad (x = y) \text{ or } (x, y \in A)$$

This could also be phrased as saying: use the equivalence relation associated to the partition of X

$$\{A\} \cup \{\{x\} \mid x \in X \setminus A\}.$$

Write X/A for the subsequent quotient space.

Example 3.23. Let $[0,1]$ be the unit interval. We claim that

$$[0,1]/\{0,1\} \cong S^1.$$

To see this, consider the q map $f\colon [0,1] \to S^1$ given by $f(x) = (\cos(2\pi x), \sin(2\pi x))$ descends to the quotient, because $f(0) = f(1)$. It is also surjective. It is injective except at $x \in \{0,1\}$, but $0 \sim 1$, so the induced map on the quotient is injective. Thus the induced map on the quotient is bijective and continuous. To show the induced map $[0,1]/\{0,1\} \to S^1$ is a homeomorphism we must show the induced map on the quotient is open. Unlike Example 3.21, the map f itself is not open, so we actually have to deal with the induced map.

Let $U \subseteq [0,1]/\{0,1\}$ be open.

(i) If $[0] \notin U$ then U consists of equivalence classes of the form $[x] = \{x\}$. So we can write $U = \{\{x\} \mid x \in V\}$ for some $V \subseteq (0,1)$. Then $q^{-1}(U) = V$, and V is an open set in $[0,1]$ (by definition of the quotient topology, using that U is open). Then V is a union of open intervals (a,b), so $f(V)$ is a union of images of open intervals. The image $f((a,b))$ is the open arc of the circle from angle $a/2\pi$ to angle $b/2\pi$, which is an open set as previously discussed. So $f(V) \subseteq S^1$, which is the image of U under the induced map, is also open.

(ii) If $[0] \in U$ then $q^{-1}(U) = V$, then V must contain both 0 and 1. We must show $f(q^{-1}(U)) \subseteq S^1$ is open. For this we argue there is an open arc around every point $(x,y) \subseteq f(q^{-1}(U)) \subseteq S^1$.

If $(x,y) \neq (0,1)$, then $f^{-1}\{(x,y)\}$ is a singleton in $V \setminus \{0,1\}$. Thus there is an open interval $(a,b) \subseteq V \setminus \{0,1\} \subseteq (0,1)$ around this point and (a,b) is mapped to an open arc in the circle, by the same argument as in (i).

If $(x,y) = (0,1)$ then $f^{-1}\{(x,y)\} = \{0,1\}$. As V is open in $[0,1]$, and contains both 0 and 1, there exists some $\varepsilon > 0$ such that $[0,\varepsilon) \cup (1-\varepsilon, 1] \subseteq V$. The image $f([0,\varepsilon) \cup (1-\varepsilon,1]) \subseteq S^1$ is the open arc from angle $-\varepsilon/2\pi$ to angle $\varepsilon/2\pi$, and so is open.

Thus $f(q^{-1}(U)) \subseteq S^1$ is open.

Thus the induced map on the quotient is open and $[0,1]/\{0,1\} \cong S^1$, as claimed.

Exercise 3.7. Write S^2 for the unit sphere in $\mathbb{R}^3$, and D^2 for the unit disc in $\mathbb{R}^2$. Using polar coordinates in $\mathbb{R}^2$, show that

$$f\colon D^2 \to S^2; \qquad f(r,\theta) = (\sin(\pi r)\cos\theta, \sin(\pi r)\sin\theta, \cos(\pi r))$$

descends to the quotient $D^2/\{r=1\}$ and that the induced map is a homeomorphism. In other words show that the quotient of the unit disc by its boundary circle is homeomorphic to the 2-sphere.

Example 3.24. The generalisation of the previous two examples is the statement that the quotient of the n-disc by its boundary sphere S^{n-1}

is homeomorphic to the n-sphere $D^n/S^{n-1} \cong S^n$. This is more easily checked by less "hands-on" methods, as we shall see later.

4 *Hausdorff spaces and limits*

Figure 4.1: Felix Hausdorff (1968 - 1942) is one of the founders of point-set topology. He became fascinated by Cantor's set theory, using it to describe the topological concepts we use today. He also coined the term "metric space".

In metric spaces, the fact that $d(x,y) = 0$ implies $x = y$ can be thought of as a saying "metric topologies are strong enough to separate points". This is not something every topology can do; for example the indiscrete topology $\tau = \{\emptyset, X\}$ can only "see" the whole set or nothing, and so does not know how to separate individual points $x, y \in X$. If a topology can separate points, we are able to recover many more of the familiar properties of a metric space.

Hausdorff spaces

Definition 4.1. A topological space (X, τ) is called *Hausdorff* if, for every $x, y \in X$, with $x \neq y$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.

Exercise 4.1. A topology induced by a metric is Hausdorff.

Exercise 4.2. Let (X, τ) be a topological space. Show that a set $A \subseteq X$ is open if and only if for every $x \in A$, there exists an open set $U \subseteq A$ with $x \in U$.

Theorem 4.2. *Let (X, τ) be a topological space. If (X, τ) is Hausdorff then $\{x\}$ is closed for every $x \in X$.*

The converse of this theorem is not true, as the next exercise shows.

Exercise 4.3 (Cofinite topology). Let X be a non-finite set. Define a topology τ on X by declaring $U \subseteq X$ to be open if $E = \emptyset$ or $X \setminus E$ is finite (you should check this is a topology). Show that every singleton $\{x\}$ is closed, but that (X, τ) is not Hausdorff.

Exercise 4.4. Let (X, τ) be a topological space. A subset $S \subseteq X$ is called *discrete* if the subspace topology on S is the discrete topology. Show that in a Hausdorff space X, every finite subset is discrete.

Lemma 4.3. *Let (X, τ) be a Hausdorff topological space and $A \subseteq X$. Then A with the subspace topology is Hausdorff.*

Lemma 4.4. *Let (X, τ) and (Y, σ) be Hausdorff topological spaces. Then $X \times Y$ with the product topology is Hausdorff.*

However, quotient spaces of Hausdorff spaces *might not* be Hausdorff! There are several classic examples of this.

Exercise 4.5 (The line with two origins). The real line $\mathbb{R}$ is a metric space, so is Hausdorff. Write $\mathbb{R}_1 = \mathbb{R} \times \{1\}$ and $\mathbb{R}_2 = \mathbb{R} \times \{2\}$. The disjoint union $X = \mathbb{R}_1 \cup \mathbb{R}_2$ is a metric space, and so is Hausdorff. Define an equivalence relation on X by $(a,b) \sim (c,d)$ if $(a,b) = (c,d)$, and also $(x,1) \sim (x,2)$ for all $x \neq 0$. The quotient space $X/\sim$ is called *the line with two origins*.
Show that the line with two origins is not Hausdorff, by showing that every open set $U \subseteq X/\sim$ around $[(0,1)]$ intersects every open set $V \subseteq X/\sim$ around $[(0,2)]$ nontrivially.

Exercise 4.6. On $\mathbb{R}$, consider the equivalence relation $x \sim y \iff x - y \in \mathbb{Q}$.

(i) Show that $\mathbb{R}/\sim$ is uncountable.

(ii) Show that the quotient topology is the indiscrete topology. (In particular, NOT Hausdorff!)

Exercise 4.7. A topological space is Hausdorff iff the diagonal

$$\Delta = \{(x,x) \mid x \in X\} \subseteq X \times X$$

is a closed subset of $X \times X$.

Note that for any set X, the relation xRy iff $x = y$ results in $X/R = X$. So Exercise 4.7 is the special case of Exercise 4.8 where $R = \Delta$.

Exercise 4.8. Suppose X is a Hausdorff space with an equivalence relation R, and that the quotient map $q \colon X \to X/R$ is open. Show that X/R is Hausdorff iff $R \subseteq X \times X$ is closed.

Neighbourhoods and limits

The idea of the limit of a sequence transferred very well from Analysis to metric spaces (Definition 1.19). We will now attempt to transfer this definition to the context of topological spaces. In general, the idea has some issues, but for Hausdorff spaces it behaves as you would hope.

Definition 4.5. Let $x \in X$ be a point in a topological space. A set $A \subseteq X$ is called a *neighbourhood* of x if there is an open set U such that $x \in U \subseteq A$.

Definition 4.6. Let $(x_n)_{n=0}^{\infty}$ be a sequence in a topological space (X, τ). Then we say $x \in X$ is a *limit of the sequence* if for every neighbourhood A of x there exists $N \geq 0$ such that $x_n \in A$ for all $n \geq N$.
We use the usual notation: "$x_n \to x$ as $n \to \infty$" or "$\lim_{n\to\infty} x_n = x$".

The following example means we should be wary of limits in general topological spaces.

Example 4.7. Let $X = \{a, b\}$, where $a \neq b$ and let τ be the indiscrete topology. Let (x_n) be any sequence in X. Then as the only neighbourhood of a is X and the only neighbourhood of b is X, it is easy to see that $x_n \to a$ and $x_n \to b$.

One of the pleasant features of a Hausdorff space is that this strange scenario cannot arise.

Theorem 4.8. *A convergent sequence in a Hausdorff space has a unique limit.*

Finally, limits of sequences give another way to think about the closure of a subset, under certain conditions.

Proposition 4.9. *Let (X, τ) be a topological space and $A \subseteq X$. Then if x is a limit of a sequence in A then it belongs to the closure of A.*

The converse is not necessarily true. That is, there exist topological spaces with subsets $A \subseteq X$ such that $x \in \mathrm{cl}(A)$ but there is no sequence in A converging to x. To fix this, we have to put more structure on the topological space.

Definition 4.10. Let (X, τ) be a topological space.

(i) X is *first countable* if for each $x \in X$, there exists a sequence of open neighbourhoods $U_1, U_2, U_3, \ldots$ of x such that for every neighbourhood N of x, we have $U_n \subseteq N$, for some $n \geq 1$.

(ii) X is *second countable* if it admits a basis with countably many elements.

Exercise 4.9. Prove second countable implies first countable.

Exercise 4.10. Prove that every metric space is first countable.

Exercise 4.11. Prove that Euclidean space is second countable.

Exercise 4.12. Prove that if X is first countable and $A \subseteq X$, then for all $x \in \mathrm{cl}(A)$, there exists a sequence $(x_n)_{n=1}^{\infty}$ in A such that $x_n \to x$.

Corollary 4.11. *Let (X, τ) be a first countable space, and $A \subseteq X$. Then A is closed if and only if the limits of all convergent sequences in A are elements of A.*

Proof. Recall that A is closed if and only if $A = \mathrm{cl}(A)$. Now combine Proposition 4.9 and Exercise 4.12. □

There are many examples of spaces that are not first countable (and thus not metrisable!). Here is a particularly cute one.

Example 4.12. Consider the crush space $\mathbb{R}/\mathbb{Z}$, crushing every integer to a single point, but leaving the non-integer points alone. You can think of this as a flower with countably infinitely many petals. There is no way to choose open neighbourhoods $U_1, U_2, U_3, \ldots$ in the quotient topology, around the class $[0]$, such that every neighbourhood of $[0]$ is contained in one of the U_n. Something like Cantor's diagonal argument is usually invoked to prove this!

5 Compactness

Compact spaces

In Analysis one defines a subset $A \subseteq \mathbb{R}^n$ to be compact if every sequence in A has a convergent subsequence. As we saw earlier, the concept of limit does not transfer very well to general topological spaces. But it turns out that the concept of compactness is fundamental in topology. To give a definition of compactness without using sequences, we need the language of *covers*.

Definition 5.1. Given a set X, a *cover of* X is collection of sets

$$\mathscr{A} = \{A_\alpha \subseteq X \mid \alpha \in I\}$$

such that $\bigcup_{\alpha \in I} A_\alpha = X$. If X has a topology and A_α is open for all $\alpha \in A$, we call $\mathscr{A}$ an *open cover*.

Definition 5.2. A topological space (X, τ) is *compact* if every open cover of X has a finite subcover.

In other words, X is compact if for any open cover $\{U_\alpha \mid \alpha \in I\}$ of X, there exists a finite subcollection $\{U_{\alpha(1)}, U_{\alpha(2)}, \ldots, U_{\alpha(n)}\}$ such that $\bigcup_{i=1}^n U_{\alpha(i)} = X$.

Definition 5.3. A subset $Y \subseteq X$ of a topological space is *compact* if Y with the subspace topology is a compact topological space.

These definitions combine with Lemma 3.2 to give the following.

Exercise 5.1. A subset $Y \subseteq X$ of a topological space is compact if for every collection of open sets $\{U_\alpha \subseteq X \mid \alpha \in I\}$ such that $Y \subseteq \bigcup_{\alpha \in I} U_\alpha$, there exists a finite subcollection $\{U_{\alpha(1)}, U_{\alpha(2)}, \ldots, U_{\alpha(n)}\}$ such that $Y \subseteq \bigcup_{i=1}^n U_{\alpha(i)}$.

Exercise 5.2. Prove the following.

(i) If X is finite, every topology on X is compact.

(ii) The set X with the discrete topology is compact if and only if X is finite.

(iii) The indiscrete topology is always compact.

Theorem 5.4. *If X is a compact space and $E \subseteq X$ is closed then E is compact.*

Theorem 5.5. *If X is Hausdorff then all compact sets are closed.*

The famous characterisation of compact sets in Euclidean space by Heine-Borel does not entirely become trivial with these new tools, but does become a lot easier. We still have to prove the following essentially "bare hands", but then the rest is easy.

Theorem 5.6 (Heine-Borel)**.** *Let $a, b \in \mathbb{R}$ with $a < b$. Then the interval $[a, b]$ is compact, using the usual topology on $\mathbb{R}$.*

Corollary 5.7. *Let $\mathbb{R}$ have its usual topology. Then a subset is compact if and only if it is closed and bounded.*

"If E's closed and bounded,
Says Heine–Borel,
And also Euclidean,
Then we can tell
That, if it we smother
With a large open cover,
There's a finite refinement as well."
– Conway

Bases, products and quotients

We consider how compactness interact with bases.

Theorem 5.8. *Let (X, τ) be a topological space with basis $\mathscr{B}$. Then X is compact if and only if every open cover of X by sets in $\mathscr{B}$ admits a finite subcover.*

We consider how compactness interacts with products.

Theorem 5.9 (Tychonoff's Theorem: finite products)**.** *If X and Y are compact, then $X \times Y$ is compact.*

We consider how compactness interacts with quotient spaces.

Theorem 5.10. *Let (X, τ) be a topological space and $\sim$ be an equivalence relation on X. Then if X is compact, so is the quotient space $X/\sim$.*

This is perhaps the most important theorem concerning compact sets.

Theorem 5.11. *The image of a compact set under a continuous map is compact.*

There are many consequences of this excellent theorem.

Theorem 5.12. *Let $\mathbb{R}$ have the usual topology, suppose $K \subseteq \mathbb{R}$ is closed and bounded and let $f \colon K \to \mathbb{R}$ be continuous. Then $f(K)$ is closed and bounded. Moreover, f attains its bounds.*

Theorem 5.13. *A continuous bijective function $f \colon X \to Y$ from a compact space to a Hausdorff space is a homeomorphism.*

Example 5.14. There is no homeomorphism between a circle and the real line.

Theorem 5.15. *Let (X, τ) be a compact topological space and $\sim$ be an equivalence relation on X. Suppose $f\colon X \to Y$ is a continuous function to a Hausdorff space Y, that descends to a bijective function $X/\sim \to Y$. Then the induced function is a homeomorphism $X/\sim \cong Y$*

Exercise 5.3. Let $n \geq 1$. Consider the continuous function

$$\begin{array}{rcl} f\colon D^n & \to & \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} \\ x & \mapsto & \begin{cases} \left(\sin(\pi||x||) \cdot \frac{x}{||x||}, \cos(\pi||x||)\right) & \text{if } x \neq 0 \\ (0,1) & \text{if } x = 0 \end{cases} \end{array}$$

(i) Prove f has image S^n (and is therefore a continuous surjective map $f\colon D^n \to S^n$).

(ii) Prove f descends to the quotient D^n/S^{n-1}.

(iii) Prove the induced map on the quotient is injective.

(iv) Prove that $D^n/S^{n-1} \cong S^n$.

Sequential compactness

We have already seen that the notion of limit of a sequence only really becomes useful when we are working in a Hausdorff space, so that limits are unique (Theorem 4.8). This means we should be fairly suspicious of the following compactness definition from Analysis.

Definition 5.16. A topological space (X, τ) is *sequentially compact* if every sequence $(x_n)_{n=0}^{\infty}$ in X has a convergent subsequence.

Adding the adjective "Hausdorff" at least makes limits unique, but it is not strong enough to ensure this definition agrees with the definition of compactness. In metric spaces, it turns out the definitions do agree.

Theorem 5.17. *Let (X, d) be a metric space. Then X with the induced topology is compact if and only if it is sequentially compact.*

In general, there are spaces that are compact but not sequentially compact, and there are spaces that are sequentially compact but not compact! Theorem 5.17 shows that adding the adjective "metrisable" to the topology is enough to make these notions agree.

One-point compactification

Compact spaces have a lot of technical advantages over non-compact spaces. In this section we discuss a technical procedure for artificially "compactifying" a space.

Construction 5.18 (One-point compactification). Let (X, τ) be a topological space. Define the *one-point compactification* (X^*, τ^*) as follows. Let

$$X^* = X \cup \{\infty\}.$$

Here "∞" is a stand-in symbol for "an object not in the set X already". We then define

$$\tau^* = \tau \cup \{K^c \cup \{\infty\} \mid K \subseteq X \text{ closed and compact}\}.$$

In other words, there are two types of open sets in X^*, the ones that were already the open sets in X, and the new ones that are the complements of closed compact sets, together with the infinity point.

Of course, we must check we have achieved the goal of compactifying!

Theorem 5.19. *Given a topological space (X, τ):*

(i) τ^* *is a topology on* X^**;*

(ii) τ^* *is a compact topology;*

(iii) the subspace topology on $X \subseteq X^*$*, coming from* τ^* *is* τ*.*

Example 5.20. If $(X, \tau) \cong (Y, \sigma)$ then $(X^*, tau^*) \cong (Y^*, \sigma^*)$.

For some applications, it would be useful to conclude that (X^*, τ^*) is the *unique* one-point compactification of (X, τ). For this we need to make some more assumptions about (X, τ).

Definition 5.21. A space X is *locally compact* if every point $x \in X$ has an open neighbourhood U such that $\mathrm{cl}(U)$ is compact.

Example 5.22. (i) Of course, compact spaces are locally compact!

(ii) Euclidean space $\mathbb{R}^n$ is the main example of a locally compact space. Each point $x \in \mathbb{R}^n$ has an open ball around it, for example $B(x; 1)$. The closure of the ball is compact by the Heine-Borel theorem.

(iii) In fact it is only important that the space X is *locally Euclidean*. That is, for every point $x \in X$, there is an open neighbourhood V of x such that there is a homeomorphism $f : V \to \mathbb{R}^n$ for some n. Then the set $f^{-1}(B(f(x); 1)) \subseteq X$ is the required neighbourhood with compact closure.

An example of a locally Euclidean space (that is not $\mathbb{R}^n$) is the n-sphere S^n.

Theorem 5.23. *If (X, τ) is a locally compact Hausdorff space, then τ^* is the unique Hausdorff topology on X^* satisfying the properties (i), (ii), and (iii) of Theorem 5.19.*

Corollary 5.24. *Let (Y, σ) be a compact Hausdorff space. Let $y \in Y$ and write $X = Y \setminus \{y\}$, and τ the subspace topology on X. If (X, τ) is locally compact, then $(X^*, \tau^*) \cong (Y, \sigma)$.*

If we remove a point from a compact space, and then point-compactify, we would like to get back to the space we started with. Corollary 5.24 gives conditions where this holds.

Example 5.25. The n-sphere S^n is compact and Hausdorff. The space $S^n \setminus \{y\}$, where $y = (0, \ldots, 0, -1)$ is the South pole, is locally-compact and Hausdorff. Thus $(S^n \setminus \{y\})^* \cong S^n$. Using the function from Exercise 5.3, one can prove that the open unit ball $B(0;1) \subseteq \mathbb{R}^n$ is homeomorphic to $S^n \setminus \{y\}$. By Example 5.20, we thus have that

$$B(0;1)^* \cong (S^n \setminus \{y\})^*.$$

Putting this all together, the one-point compactification of the open n-disc is the n-sphere!

6 Connectedness

Suppose that U is an open subset of $\mathbb{R}$ (in the usual topology) and $f\colon U \to \mathbb{R}$ is a differentiable function with $f'(u) = 0$ for all $u \in U$. Surely, we can conclude that f is constant? No! We are not being careful enough.

For example, we can set $U = (0,1) \cup (2,3)$, and define

$$f(u) = \begin{cases} 0 & \text{if } 0 < u < 1, \\ 1 & \text{if } 2 < u < 3. \end{cases}$$

This satisfies all the conditions, but is not constant.

What extra condition should we put on U to make the result true?

Connected spaces

Definition 6.1. A topological space (Y, σ) is said to be *disconnected* if we can find non-empty open sets U and V such that $U \cup V = Y$ and $U \cap V = \emptyset$. A space which is not disconnected is called *connected*.

Definition 6.2. If E is a subset of a topological space (X, τ) then E is called *connected* (respectively *disconnected*) if the subspace topology on E is connected (respectively disconnected).

The definition of a subspace topology gives the following alternative characterisation which the reader may prefer.

Lemma 6.3. *If E is a subset of a topological space (X, τ), then E is disconnected if and only if we can find open sets U and V such that $U \cup V \supseteq E$, $U \cap V \cap E = \emptyset$, $U \cap E \neq \emptyset$ and $V \cap E \neq \emptyset$*

Here is another alternative characterisation which shows that we are on the right track.

Theorem 6.4. *If E is a subset of a topological space (X, τ), then E is disconnected if and only if we can find a non-constant continuous function $f\colon E \to \{0,1\}$, where $\{0,1\}$ is given the discrete topology.*

Here is another (even easier) observation that is pretty useful in practice.

Lemma 6.5. *A space X is disconnected if and only if it contains a proper subset $A \subseteq X$ that is both closed and open.*

Exercise 6.1. Show that the following sets A are disconnected:

(i) $A = \{0,1\} \subseteq \mathbb{R}$

(ii) $A = [-1,0) \cup (0,1] \subseteq \mathbb{R}$

(iii) $A = \mathbb{Q} \subseteq \mathbb{R}$

(iv) $A = A_1 \cup A_2 \subseteq \mathbb{R}^2$ where $A_1 = \mathbb{R} \times \{0\}$ and $A_2 = \{(x,y) \mid xy = 1\}$.

Proposition 6.6. *If $A \subseteq X$ is connected and $A \subseteq B \subseteq \mathrm{cl}(A)$, then B is connected. In particular, the closure of a connected set is connected.*

In other words, if a set is connected it is still connected after you add any number of its limit points.

The following deep result is now easy to prove (if we are willing to use the Intermediate Value Theorem in our proof).

Theorem 6.7. *If we give $\mathbb{R}$ the usual topology, then $\mathbb{R}$ and the intervals $[a,b]$ and (a,b) are connected.*

Exercise 6.2.

(i) If (X,τ) and (Y,σ) are topological spaces, E is a connected subset of X and $g : E \to Y$ is continuous, then $g(E)$ is connected.

"The continuous image of a connected set is connected."

(ii) If (X,τ) is a connected topological space and $\sim$ is an equivalence relation on X, then $X/\sim$ with the quotient topology is connected.

(iii) If (X,τ) and (Y,σ) are connected topological spaces, then $X \times Y$ with the product topology is connected.

(iv) If (X,τ) is a connected topological space and E is a subset of X, it does not follow that E with the subspace topology is connected.

The proof of the next example is particularly important because it gives a standard technique for using connectedness in practice.

Example 6.8. Suppose that E is a connected subset of a topological space (X,τ). Suppose that $f : E \to \mathbb{R}$ is "locally constant" in the sense that, given any $e \in E$, we can find an open neighbourhood U of e such that f is constant on $U \cap E$. Then f is constant.

The connected components

"The union of a collection of connected subsets that have a point in common is connected."

Lemma 6.9. *Let $\{A_\alpha \mid \alpha \in I\}$ be a collection of connected subsets of a topological space, and suppose there exists $x \in X$ such that $x \in A_\alpha$ for all $\alpha \in I$. Then the union $\bigcup_{\alpha \in I} A_\alpha$ is connected.*

The previous lemma ensures the following definition results in a connected set.

Definition 6.10. Let (X,τ) be a topological space and let $x \in X$. Then the *connected component* of x is the union over all connected sets containing x.

Exercise 6.3. Let (X,τ) be a topological space and for $x,y \in X$, write $x \sim y$ if there exists a connected set $E \subseteq X$ such that $x,y \in E$. Prove that $\sim$ is an equivalence relation on X and that $[x]$ is the connected component of x.

Exercise 6.4.

(i) Prove that a homeomorphism $f\colon A \to B$ induces a bijection between the sets of connected components. In particular, this shows the cardinality of the set of connected components is a topological invariant.

(ii) Prove that if $f\colon X \to Y$ is a homeomorphism then for any point $x \in X$ the restriction to the subspaces $f\colon X \setminus \{x\} \to Y \setminus \{f(x)\}$ is a homeomorphism.

(iii) Use the previous two results to classify the following capital letters of the alphabet up to homeomorphism:

The letters are 1-dimensional closed subsets of $\mathbb{R}^2$. Also, use this font!

A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z

Connectivity in the reals

In some sense, proving that $[a,b]$ in connected by using the Intermediate Value Theorem is cheating and not in the spirit of point-set topology because most proofs of the IVT from Analysis essentially invoke the connectedness of the interval. Let's prove it again, without this crutch.

Theorem 6.11. *If we give $\mathbb{R}$ the usual topology, then $\mathbb{R}$ and the intervals $[a,b]$ and (a,b) are connected.*

Proof. Consider $[a,b]$ and suppose for a contradiction that there exists a proper subset $A \subseteq [a,b]$ that is both open and closed. By replacing A by its complement (which is also proper, open and closed), if necessary, we can assume $b \notin A$. As A is a closed subset of a compact space, it is compact and thus has a maximal element $t \in A$. As A is open there exists $B(t;\delta) \subseteq A$. As $t < b$, there exists some $s \in B(t;\delta)$ with $t < s < b$. But then $s \in A$ and $t < s$. Contradiction.

The version for (a,b) then follows by observing that (a,b) is the union of closed intervals around $(b-a)/2$, the centre point[1]. Now apply Lemma 6.9. □

[1] E.g. $(-1,1) = \bigcup_{n\in\mathbb{N}}[-1+\frac{1}{n}, 1-\frac{1}{n}]$.

Using this, we can get a souped up IVT.

Theorem 6.12 (Intermediate Value Theorem)**.** *Let $f\colon X \to \mathbb{R}$ be a continuous map from a connected space X, and let $a, b \in X$. Then for all $y \in \mathbb{R}$ with $f(a) < y < f(b)$, there exists $c \in X$ with $f(c) = y$.*

Proof. Consider

$$U = f(X) \cap (-\infty, y) \qquad \text{and} \qquad V = f(X) \cap (y, \infty).$$

As X is connected and f is continuous, $f(X)$ is connected. As $f(a) \in U$ and $f(b) \in V$, these are non-empty open subsets of the connected set $f(X)$. This implies they do not cover $f(X)$. Hence $y \in f(X)$ and the result follows. □

Exercise 6.5. Prove that if $f\colon S^1 \to \mathbb{R}$ is a continuous function then there exists $x \in S^1$ such that $f(x) = f(-x)$.

Exercise 6.5 works in greater generality. Suppose X is a topological space with an *involution*; that is a continuous function $I\colon X \to X$ such that $I \circ I = \mathrm{Id}_X$ (e.g. a reflection). Then there exists $x \in X$ such that $f(x) = f(I(x))$.

Exercise 6.6. Suppose $f\colon [0,1] \to [0,1]$ is a continuous function. Then there exists $x \in [0,1]$ such that $f(x) = x$.

Path connectivity, local connectivity, and path components

Connectedness is related to the following, older, concept.

Definition 6.13. A *path* in a topological space (X, τ) is a continuous function

$$\gamma\colon [0,1] \to X.$$

Writing $\gamma(0) = a$ and $\gamma(1) = b$, we say the path is *from a to b*.
We say (X, τ) is *path-connected* if, for all $a, b \in X$ there exists a path in X from a to b.

Exercise 6.7. Show that the following sets are path connected by writing down paths between any two arbitrary points.

(i) $\mathbb{R}^n$

(ii) $B(x; r) \subseteq \mathbb{R}^n$ for all $x \in \mathbb{R}^n$ and all $r > 0$.

(iii) $\mathbb{R}^n \setminus \{0\}$ "punctured Euclidean space"

(iv) S^n for $n > 0$.[2]

[2] *Hint: it is easy to prove that the continuous image of a path-connected space is path connected (prove this). Express S^n as such a continuous image.*

Construction 6.14. Suppose we have two paths

$$\gamma\colon [0,1] \to X \quad \text{and} \quad \delta\colon [0,1] \to X$$

such that $\gamma(1) = \delta(0)$. Then we can *concatenate* the paths as follows, to define a new path from $\gamma(0)$ to $\delta(1)$.

$$\gamma \bullet \delta\colon [0,1] \to X; \qquad t \mapsto \begin{cases} \gamma(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \delta(2t-1)) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

Concatenated paths are still paths (i.e. are continuous functions), due to the following lemma.

Lemma 6.15. *Suppose* $X = \bigcup_{i=1}^{n} E_i$ *is a finite union of closed subsets* $E_i \subseteq X$*. If a function* $f\colon X \to Y$ *such that for all* $i = 1, \ldots, n$*, the restriction* $f|_{E_i}\colon E_i \to Y$ *is continuous, then* f *is continuous,*

The operation of concatenation should help you prove transitivity in the following.

Exercise 6.8. Let (X, τ) be a topological space and for $x, y \in X$ write $x \sim y$ when there is a path in X from x to y. Show that $\sim$ is an equivalence relation.

Definition 6.16. Let (X, τ) be a topological space. The equivalence classes under the equivalence relation "there is a path from x to y" are called the *path-components* of X.

Write $\pi_0(X)$ for the set of path components of X.

The following is fairly easy to prove.

Theorem 6.17. *Let* (X, τ) *be a topological space. If* X *is path-connected then* X *is connected.*

The converse is generally false. You might know this example from an earlier Analysis class. It is frequently used as a counterexample to spurious statements.

Exercise 6.9 (The topologist's sine curve). Write

$$E = \{(x, \sin(1/x)) \in \mathbb{R}^2 \mid 0 < x < 1)\}$$

for the graph of the *topologist's sine curve*, and write

$$F = \{(0, y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\}.$$

Show that the union $E \cup F \subseteq \mathbb{R}^2$ is connected, but not path connected.

The converse is true when we add in an extra condition.

Definition 6.18. A space X is *locally path connected* if every $x \in X$ has an open path connected neighbourhood.

Theorem 6.19. *If* X *is connected and locally path connected then* X *is path connected.*

Proof. Let $x \in X$. Suppose X is connected and locally path connected but not connected. We will show there exists a proper subset that is both open and closed (a contradiction to being connected). Let

$$E = \{y \in X \mid \text{there is a path from } x \text{ to } y\}.$$

As $x \in E$, this set is non-empty. As X is not path connected, this set is proper.

We show E is open. Let $y \in E$. As X is locally path connected, y has an open path-connected neighbourhood N. For all $z \in N$, there is a path from y to z. Concatenating this with a path from x to y, there is a path from x to z. So $N \subseteq E$ and hence E is open.

We show E^c is open. If $y \in E^c$ then there is an open, path connected neighbourhood N of y. If there is some element $z \in N \cap E$, we could concatenate a path from x to z with a path from z to y to get a path from x to y, which does not exist. Hence $N \subseteq E^c$ and so E^c is open.

We have shown E is both open and closed, which is a contradiction as X is connected. Thus X is path connected. □

Example 6.20. Any open, connected subset U of $\mathbb{R}^n$ is path connected.

7 The fundamental group

We now switch from *point-set topology* into *Algebraic Topology*.

Convention: from now on we will use the word *map* to mean "continuous function". This is to prevent having to write the word "continuous" all the time. This is a very common (but not universal) convention in Topology.

Convention: from now on, X and Y will always denote topological spaces and the topology will be suppressed from the notation.

The overarching idea of Algebraic Topology is to study topological spaces by extracting cruder, simpler quantities from them: we seek to assign discrete, algebraic objects to topological spaces. These provide a way of distinguishing and, hopefully, characterising the space. These algebraic objects should be *invariant* under continuously "deforming" the topology. The precise meaning of "deformation" in this context is given by the concept of *homotopy*, the theme of this section.

We begin by discussing paths in our space, up to deformation.

Homotopy of paths

Notation: $I = [0,1]$ and $\partial I = \{0,1\}$ is now a standing notation.

Write $I = [0,1]$ and $\partial I = \{0,1\}$. A path from a to b in X is a map

$$\gamma\colon I \to X$$

with $\gamma(0) = a$ and $\gamma(1) = b$. We now describe what it means to "deform" such a path continuously into another path.

Definition 7.1. Let $a, b \in X$ and suppose $\gamma, \delta\colon I \to X$ are paths from a to b. A *homotopy from γ to δ relative to ∂I* is a map

$$F\colon I \times [0,1] \to X$$

such that:

- $F(s,0) = \gamma(s)$ and $F(s,1) = \delta(s)$ for all $s \in I$;
- $F(0,t) = a$ and $F(1,t) = b$ for all $t \in [0,1]$.

If there exists such an F, we write $\gamma \simeq \delta$ rel. ∂I.

Remark 7.2. You should think of a homotopy from γ to δ as 1-parameter family of paths, where each path in the family starts at a and ends at b. It is helpful to think of the t co-ordinate as "time". Then at each time $F(-,t)\colon I \to Y$ is a path. At time 0 we have γ and at time 1 we have δ. This captures the idea of deforming γ to δ, while "pinning" down the ends to a and b at all time.

Definition 7.3. A subset $X \subseteq \mathbb{R}^n$ is *convex* if for all $x, y \in X$ we have $tx + (1-t)y \in X$ for all $t \in [0,1]$.

Example 7.4. Any two paths in a convex subset $X \subseteq \mathbb{R}^n$ are homotopic rel. ∂I. To see this, let $\gamma, \delta \colon I \to X$ be paths in X from a to b in X. Then each point $\gamma(s)$ can by joined to the corresponding point $\delta(s)$ be a straight line. We then deform γ to δ pushing along these lines all at once:

$$F \colon I \times [0,1] \to X; \qquad F(s,t) = (1-t)\gamma(s) + t\delta(s).$$

The function F is continuous by Exercise 7.1, below.

Exercise 7.1. If $f \colon X \to \mathbb{R}^n$, $g \colon Y \to \mathbb{R}^n$ and $h \colon Z \to \mathbb{R}$ are continuous functions, then the following functions are continuous:

- $X \times Y \to \mathbb{R}^n; \qquad (x,y) \mapsto f(x) + g(y)$
- $X \times Z \to \mathbb{R}^n; \qquad (x,y) \mapsto f(x)h(z)$

Definition 7.5. Let $\gamma \colon I \to X$ be a path from a to b in X. Then the *inverse* path is the path from b to a in X given by

$$\gamma^{-1} \colon I \to X; \qquad \gamma^{-1}(t) = \gamma(1-t).$$

Proposition 7.6. *Let $a, b \in X$. The relation $\gamma \simeq \delta$ rel. ∂I is an equivalence relation on the set of paths from a to b.*

General paths are important, but we will mainly be interested in paths that start and end at the same point.

Definition 7.7. We call a path $\gamma \colon I \to X$ a *loop* if $\gamma(0) = \gamma(1)$. We say the loop is *based at* $\gamma(0)$.

Definition 7.8. Fix $x_0 \in X$. Define $\pi_1(X, x_0)$ to be the set of equivalence classes of loops in X under the relation that the loops are homotopic rel. boundary.

In fact this set of equivalence classes forms a group, as we show in the next section.

The fundamental group

We will now show that $\pi_1(X, x_0)$ is a group. Groups need an identity element, so we define this now.

Definition 7.9. Given $x \in X$, define the *constant path* to be $c_x \colon I \to X$ given by $c_x(t) = x$ for all $t \in I$.

Proposition 7.10. *Write $[\gamma]$ for the equivalence class of a path γ under the relation of homotopy rel. boundary.*

(i) Show that the operation $[\gamma] \bullet [\delta] := [\gamma \bullet \delta]$ *is well-defined.*

(ii) Show that this operation is associative and unital.

(iii) Show that $[\gamma \bullet \gamma^{-1}] = [c_{\gamma(0)}]$.

This exercise proves the following.

Theorem 7.11. *Let* $x_0 \in X$. *Then* $\pi_1(X, x_0)$ *is a group under the operation of path concatenation, with identity element* $[c_{x_0}]$ *and inverses* $[\gamma]^{-1} = [\gamma^{-1}]$. *This is called the* fundamental group *of* X.

Dependence on the basepoint

The reader will have spotted the extra choice that had to be made in order to define the fundamental group, namely the basepoint $x_0 \in X$. If two choices of basepoint are connected by a path, the resulting fundamental groups are related as follows.

Theorem 7.12. *Let* $x_0, y_0 \in X$ *and suppose* α *is a path from* x_0 *to* y_0. *Then the map*

$$C_\alpha \colon \pi_1(X, x_0) \to \pi_1(X, y_0); \qquad C_\alpha([\gamma]) = [\alpha^{-1} \bullet \gamma \bullet \alpha]$$

is an isomorphism, with inverse $C_{\alpha^{-1}}$.

Thus if X is a path-connected space, the *isomorphism class* of the fundamental group, but not necessarily the group itself, is well-defined.[1]

Remark 7.13. If α and β are two paths from x_0 to y_0, then there is a self-isomorphism

$$C_{\beta^{-1}} \circ C_\alpha \colon \pi_1(X, x_0) \to \pi_1(X, x_0); \;\; [\gamma] \mapsto [(\alpha \bullet \beta^{-1})^{-1} \bullet \gamma \bullet (\alpha \bullet \beta^{-1})]$$

This shows the fundamental group of a path-connected space is well-defined, not just up to arbitrary isomorphism, but more specifically up to conjugation. In particular, if the fundamental group is abelian then conjugation is the identity map and the fundamental group is well-defined independent of basepoint.

Definition 7.14. A space X is *simply connected* if it is path connected and has trivial fundamental group.[2] In other words, X is simply connected if $\pi_0(X)$ and $\pi_1(X)$ are both singletons.

Homotopy

The fundamental group of a space is a fairly robust invariant. Not only is it preserved under homeomorphisms of the space, it is preserved under the much more violent idea of "deforming" the entire space. We now make precise what it means to deform a general map and to deform a space itself.

[1] It is *very* common for people to gloss over this point and simply write $\pi_1(X)$ when X is a path-connected space, indicating the reader is free to choose their own basepoint because the author only cares about the isomorphism class of the group.

[2] Though fundamental groups generally depend on the basepoint, the property of having trivial fundamental group is well-defined in a path connected space.

Definition 7.15. Let $f, g\colon X \to Y$ be maps. A *homotopy* from f to g is a map

$$H\colon X \times [0,1] \to Y$$

such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$.
We write $f \simeq g$, and say f *is homotopic to* g if there exists a homotopy from f to g. To specify a homotopy, we can write $H\colon f \simeq g$.

You should think of a homotopy from f to g as 1-parameter family of functions $H(-,t)\colon X \to Y$, starting at f when $t = 0$ and ending at g when $t = 1$. This captures the idea of deforming f to g.

Example 7.16 (Linear interpolation)**.** Let $f, g\colon X \to \mathbb{R}^n$ be continuous maps. Then the map

$$H\colon X \times [0,1] \to \mathbb{R}^n; \qquad H(x,t) = (1-t)f(x) + tg(x)$$

is a linear combination of continuous functions, thus continuous overall, by Exercise 7.1. Thus it is a homotopy from f to g. More generally, this proof would work whenever $f, g\colon X \to Y$, where $Y \subseteq \mathbb{R}^n$ is convex.

Suppose $f, g, h\colon X \to Y$ are maps, that $H\colon f \simeq g$ and $J\colon g \simeq h$. Then we can *concatenate* the homotopies as follows, to define a homotopy $H \bullet J\colon f \simeq h$.

Our convention in these notes is that concatenation reads left-to-right. There is no agreed upon convention for this, and other texts use right-to-left.

$$H \bullet J\colon X \times [0,1] \to Y; \qquad (x,t) \mapsto \begin{cases} H(x,2t) & \text{if } 0 \le t \le \frac{1}{2} \\ J(x,2t-1)) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

This operation should help you prove transitivity in the following.

Exercise 7.2. Prove that the relation "f is homotopic to g" is an equivalence relation on the set of maps from X to Y.

The notion of homotopic maps is used to describe the idea of homotopy equivalent spaces.

Definition 7.17. A map $f\colon X \to Y$ is a *homotopy equivalence* of topological spaces if f admits a *homotopy inverse* $g\colon Y \to X$; that is, a map such that $g \circ f \simeq \mathrm{Id}_X$ and $f \circ g \simeq \mathrm{Id}_Y$.
We write $X \simeq Y$, and say X and Y are *homotopy equivalent* if there exists a homotopy equivalence. This is clearly an equivalence relation on the set of topological spaces.

If we had instead insisted the stronger condition that $g \circ f = \mathrm{Id}_X$ and $f \circ g = \mathrm{Id}_Y$, then g would be a continuous inverse and hence f would be a homeomorphism. This shows that homeomorphic spaces are also homotopy equivalent. The converse is emphatically not true – homotopy equivalence is a much cruder equivalence relation than homeomorphism.

Definition 7.18. A space that is homotopy equivalent to a singleton (with the unique topology), is called *contractible*.

The following proposition provides good intuition for contractibility.

Proposition 7.19. *A space X is contractible if and only if the identity map Id_X is homotopic to a constant map $c\colon X \to X$; $c(x) = x_0$ for some $x_0 \in X$ and all $x \in X$.*

Intuitively: the map that fixes every point in X can be deformed into the map that sends all points to some x_0. You can think of the deformation itself as the contraction.

Example 7.20 (Euclidean space is contractible)**.** Consider the homotopy $F\colon \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ given by $F(\mathbf{x}, t) = t\mathbf{x}$. For $t = 0$, this is the constant map $c\colon X \to X$, sending all points to $0 \in X$. For $t = 1$ this is the identity map Id_X. By Proposition 7.19, the space $\mathbb{R}^n$ is contractible.

Note that for all $t > 0$, $F(-,t)\colon X \to X$ is a surjective map then *suddenly*, at $t = 0$, it is not! This might seem counterintuitive, but illustrates how violent homotopies can be.

Example 7.21 (Punctured Euclidean space is a homotopy sphere)**.** Write

$$S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid ||\mathbf{x}|| = 1\}.$$

Write $f\colon S^n \hookrightarrow \mathbb{R}^n \setminus \{0\}$ for the inclusion map and write

$$g\colon \mathbb{R}^{n+1} \setminus \{0\} \to S^n; \qquad f(\mathbf{x}) = \frac{\mathbf{x}}{||\mathbf{x}||}.$$

The maps f and g are homotopy inverses of each other. One direction is easy, as $g \circ f = \mathrm{Id}_{S^n}$ without even using a homotopy. To see that $f \circ g \simeq \mathrm{Id}_{\mathbb{R}^{n+1}\setminus\{0\}}$ we use the "linear interpolation" homotopy. Namely

$$F\colon (\mathbb{R}^{n+1} \setminus \{0\}) \times [0,1] \to \mathbb{R}^{n+1} \setminus \{0\}; \qquad F(\mathbf{x}, t) = t\mathbf{x} + (1-t)\frac{\mathbf{x}}{||\mathbf{x}||}.$$

We now wish to define our first algebraic object associated to a topological space, that is invariant under homotopy equivalence of topological spaces. Namely, the number of connected components.

Exercise 7.3. Show that if $X \simeq Y$, then the respective sets of path components $\pi_0(X)$ and $\pi_0(Y)$ have the same cardinality.

Finally, when we introduced homotopy of paths we insisted that the endpoints of our paths were "pinned down" in X throughout the deformation. This notion is more generally provided by the following.

Definition 7.22. Let $A \subseteq X$ be a subset and suppose

$$F\colon X \times [0,1] \to Y$$

is a homotopy such that $F|_{A\times[0,1]}(x,t)$ is independent of t. Then we say F is a homotopy *relative to* A (or *rel.* A).

Exercise 7.4. Given a map $f\colon A \to Y$, consider the set of maps $X \to Y$ that restrict to f on A. Show that "homotopic rel. A" is an equivalence relation on this set.

Homotopy invariance of $\pi_1(X, x_0)$

We now make precise the sense in which the fundamental group of homotopy equivalent spaces are "the same".

Proposition 7.23. *Given a map $p\colon X \to Y$, the function*

$$p_*\colon \pi_1(X, x_0) \to \pi_1(Y, p(x_0)); \qquad [\gamma] \mapsto [p \circ \gamma]$$

is a well-defined group homomorphism. Furthermore, given a map $q\colon Y \to Z$, we have $(q \circ p)_* = q_* \circ p_*$.

We would like homotopy equivalent maps $p, q\colon X \to Y$ to induce the same fundamental group homomorphism. This doesn't quite make sense as stated, because homotopies do not need to fix the image of x_0 in Y, so the basepoint might be changing throughout. However, the following is true.

Proposition 7.24. *Let $p, q\colon X \to Y$ be maps and let $x \in X$. Suppose $H\colon p \simeq q$. Then the following is a commutative diagram of group homormorphisms*

$$\begin{array}{ccc} & \pi_1(X,x) & \\ {}^{p_*}\swarrow & & \searrow^{q_*} \\ \pi_1(Y,p(x)) & \xleftarrow[\cong]{C_\alpha} & \pi_1(Y,q(x)), \end{array}$$

where $\alpha(s) = H(x,s)$ is the path from $p(x)$ to $q(x)$.

In particular:

Corollary 7.25. *Let $p, q\colon X \to Y$ be maps and let $x \in X$. Suppose $H\colon p \simeq q$ rel. $\{x\}$. Then the group homomorphisms*

$$p_*, q_*\colon \pi_1(X,x) \to \pi_1(Y,p(x))$$

are equal.

A further corollary shows we have achieved our goal of showing that the fundamental group (or at least its isomorphism class) is a well-defined homotopy invariant of the space X.

Corollary 7.26. *If X and Y are homotopy equivalent, path-connected spaces, then $\pi_1(X,x_0) \cong \pi_1(Y,y_0)$ for all $x_0 \in X$ and $y_0 \in Y$.*

*Categorical language – for enthusiasts**

This is not a course in category theory, but some of the statements above can be phrased nicely in that language.

Definition 7.27. A *category* $\mathcal{C}$ consists of a "collection"[3] of *objects* and, for each pair of objects A, B, a set $\mathrm{Hom}(A,B)$ of *morphisms* $f\colon A \to B$ from A to B. There should be an *identity* morphism $\mathrm{Id}_A \in \mathrm{Hom}(A,A)$ for every object A, and a composition law

$$\circ\colon \mathrm{Hom}(B,C) \times \mathrm{Hom}(A,B) \to \mathrm{Hom}(A,C)$$

for each triple. Composition must be associative and the identity maps should function as you expect.

[3] Technically a *class*, in the sense of ZFC set theory. In many practical settings, $\mathrm{Ob}(\mathcal{C})$ is just an ordinary set, and in this case the category is called *small*.

Definition 7.28. A *functor* $F\colon \mathcal{C} \to \mathcal{D}$ is a "map" of categories, in the following sense. The functor assigns to each object A of $\mathcal{C}$ an object $F(A)$ of $\mathcal{D}$. The functor assigns to each morphism $f\colon A \to B$ in $\mathcal{D}$ a morphism $F(f)\colon F(A) \to F(B)$ in $\mathcal{D}$, such that

$$F(\mathrm{Id}_A) = \mathrm{Id}_{F(A)} \quad \text{and} \quad F(g \circ f) = F(g) \circ F(f).$$

Example 7.29. Write $\mathscr{T}$ for the category with objects topological spaces and morphisms continuous functions.

Sadly, $\mathscr{T}$ is not a small category. To see this consider that there are at least as many topological spaces as there are sets, because every set admits the discrete topology. As there is no set of all sets, the object collection of $\mathscr{T}$ is not a set.

Write $\mathscr{T}_*$ for the category with objects based topological spaces (X, x_0) and morphisms basepoint-preserving continuous functions.

Write $\mathscr{G}$ for the category whose objects are groups and whose morphisms are group homomorphisms.

With these categories in mind, we see that π_1 is a functor

$$\pi_1\colon \mathscr{T}_* \to \mathscr{G}$$

where the pointed space (X, x_0) is assigned the group $\pi_1(X, x_0)$ and map of pointed spaces $p\colon (X, x_0) \to (Y, y_0)$ is assigned

$$\pi_1(p) := p_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0).$$

In fact, more is true. The results concerning the homotopy invariance of π_1 can be phrased this way as well.

Example 7.30. Write $\mathrm{ho}\mathscr{T}$ for the *homotopy category* of $\mathscr{T}$. This is the category with the same objects as $\mathscr{T}$, but with the morphisms the homotopy classes of continuous maps.

Note, the homotopy version of a category has potentially fewer morphisms than the original, and that homotopy equivalences in the original category become isomorphisms in the homotopy category.

Similarly, write $\mathrm{ho}\mathscr{T}_*$ for the *homotopy category* of $\mathscr{T}_*$. This is the category with the same objects as $\mathscr{T}_*$, but with the morphisms the homotopy classes of basepoint-preserving continuous maps.

The homotopy invariance of π_1 shows that the functor $\pi_1\colon \mathscr{T}_* \to \mathscr{G}$ factors through the homotopy category, in the sense that it is a composition of functors

$$\mathscr{T}_* \to \mathrm{ho}\mathscr{T}_* \xrightarrow{\pi_1} \mathscr{G},$$

where the first functor in this composition sends objects to themselves and a morphism to the homotopy class of that morphism.

8 Covering maps and computations

Here are a couple of easy computations we can make right now.

Theorem 8.1. *The fundamental group of a contractible space is the trivial group.*

Proof. Let $X \simeq \{x\}$. In particular, this implies X is path-connected as cardinality of $\pi_0(X)$ is a homotopy invariant of a space. By Corollary 7.26, for any choice of $x_0 \in X$ there is an isomorphism $\pi_1(X, x_0) \cong \pi_1(\{x\}, x)$. The latter is clearly the trivial group. □

Example 8.2. We now know that $\pi_1(\mathbb{R}^n, 0) = 0$ for any $n \in \mathbb{N}$.

Unfortunately, computing fundamental groups "bare hands" like this turns out to be fairly laborious in most other situations. It quickly becomes important to have theorems for helping the process along, and these would be developed in a longer course in Algebraic Topology.

In these notes, we will rely on the idea of *covering spaces* to make a few simple computations of fundamental groups, primarily the example of $\pi_1(S^1) \cong \mathbb{Z}$.

Covering maps

Definition 8.3. Let $p\colon E \to B$ be a map. We call p a *covering map* if there is an open cover $\{U_\alpha \mid \alpha \in I\}$ of B such that for each $\alpha \in I$, the preimage $p^{-1}(U_\alpha)$ is the disjoint union of open sets, each mapped homeomorphically to U_α by p.

If $p\colon E \to B$ is a covering map then we call E a *covering space* of B, and call B the *base space* of p. For $x \in B$ we refer to $p^{-1}(\{x\})$ as the *fibre* over x.

Remark 8.4. The fibre over a point in the base of a covering map is discrete. The existence of the open cover $\{U_\alpha \mid \alpha \in I\}$ implies $x \in U_\alpha$

for some $\alpha \in I$ and there is a commuting triangle of maps

$$\begin{array}{ccc} p^{-1}(U_\alpha) & \xrightarrow{\cong} & p^{-1}(\{x\}) \times U_\alpha \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_2} & \\ & U_\alpha & \end{array}$$

Example 8.5. Let A be any set with the discrete topology and X be any space. Then the projection map $\mathrm{pr}_2\colon A \times X \to X$ is a covering map. This is called a *trivial cover*. For example $\{1,2\} \times X \to X$ is called the trivial double cover of X.

Example 8.6. Let S^1 be the unit circle in the complex plane. The map

$$p\colon S^1 \to S^1; \qquad p(z) = z^n$$

is a covering map. To see this, write

$$A(a,b) = \{e^{i\theta} \in S^1 \mid \theta \in (a,b)\}$$

We can then for example, use the open cover $\{A(0,2\pi), A(-\pi,\pi)\}$ of S^1, in which case

$$p^{-1}(A(0,2\pi)) = \bigsqcup_{k=0}^{n} A(2\pi k/n, 2\pi(k+1)/n),$$
$$p^{-1}(A(-\pi,\pi)) = \bigsqcup_{k=0}^{n} A(\pi(2k-1)/n, \pi(2k+1)/n),$$

where

$$p\colon A(2\pi k/n, 2\pi(k+1)/n) \to A(0,2\pi)$$
$$p\colon A(\pi(2k-1)/n, \pi(2k+1)/n) \to A(-\pi,\pi)$$

are homeomorphisms for all k.

Example 8.7. The map

$$p\colon \mathbb{R} \to S^1; \qquad p(x) = \exp(2\pi i x)$$

is a covering map. To see this, and continuing the notation from the previous example, use the open cover $\{A(0,2\pi), A(-\pi,\pi)\}$ of S^1, in which case

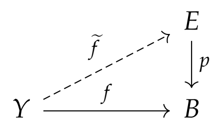
$$p^{-1}(A(0,2\pi)) = \bigsqcup_{n\in\mathbb{Z}} (n, n+1), \qquad p^{-1}(A(-\pi,\pi)s) = \bigsqcup_{n\in\mathbb{Z}} (n-\tfrac{1}{2}, n+\tfrac{1}{2}),$$

where $p\colon (n, n+1) \to A(0,2\pi)$ and $p\colon (n-\frac{1}{2}, n+\frac{1}{2}) \to A(-\pi,\pi)$ are homeomorphisms for all n.

The homotopy lifting property

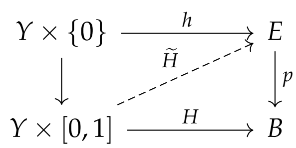
Covering maps have a strong property, which we now discuss.

Definition 8.8. Given any map of spaces $p\colon E \to B$, and another map $f\colon Y \to B$, we call a map $\widetilde{f}\colon Y \to E$ a *lift* (along p) of f if the following diagram commutes



Definition 8.9. A map $p\colon E \to B$ has the *homotopy lifting property* (HLP) if, given a homotopy $H\colon Y \times [0,1] \to B$, and a map $h\colon Y \times \{0\} \to E$ such that $p \circ h = H|_{Y\times\{0\}}$, there exists a unique map $\widetilde{H}$ making the following diagram commute.

You should read this as "once you have lifted the start of the homotopy to E, you can lift the entire thing, and in a unique way."



The next proposition

Proposition 8.10. *Let $p\colon E \to B$ be a map with the homotopy lifting property. Let $b \in B$, suppose $p^{-1}(\{b\})$ is discrete, and let $e \in p^{-1}(\{b\})$. Then the map*

$$p_*\colon \pi_1(E,e) \to \pi_1(B,b)$$

is injective. The subgroup

$$p_*(\pi_1(E,e)) \leq \pi_1(B,b)$$

is the set of homotopy classes $[\gamma]$ such that $\widetilde{\gamma}$ is a loop (rather than a path with $\widetilde{\gamma}(1) \neq e$).

Proof. Let α be a loop in E such that $p_*([\alpha]) = 1$. Then there is a homotopy $H\colon p \circ \alpha \simeq c_b$ to the constant loop c_b at the basepoint b. Lifting this homotopy $\widetilde{H}\colon I \times [0,1] \to E$, relative to $\alpha = H|_{I\times\{0\}}$, we obtain a homotopy from α to some path α' such that $p \circ \alpha'$ is the constant map to the basepoint b. As $p^{-1}(\{b\})$ is a discrete set, this implies α' is a constant map to some point $e' \in p^{-1}(\{b\})$. But considering $\widetilde{H}|_{\{0\}\times I}$ is also the unique lift of the the constant loop at the basepoint such that $\widetilde{H}|_{\{0\}\times I}(0,0) = e$, this must be the constant map to e. In particular, $\widetilde{H}|_{\{0\}\times I}(0,1) = e' = e$. So $\widetilde{H}\colon \alpha \simeq c_e$.
The identification of the subgroup is clear. □

Definition 8.11. Let $p\colon E \to B$ be a map with the homotopy lifting property. Let $e \in E$ and write $b = p(e)$. Define the *connecting map*

$$\partial_e\colon \pi_1(B,b) \to p^{-1}(\{b\})$$

by sending $[\gamma] \in \pi_1(B, b)$ to $\widetilde{\gamma}(1)$, where $\widetilde{\gamma}$ is the unique lift of γ a path in E starting at e.

Proposition 8.12. *The connecting map is well-defined.*

Proof. Suppose $H\colon \gamma \simeq \delta$ is a homotopy rel. b between loops γ and δ in B, based at b. By the homotopy lifting property, there is a unique lift of H to E, to a homotopy $\widetilde{H}$ starting at $\widetilde{\gamma}$. Consider that $H|_{\{0\}\times[0,1]}$ is the constant map to $b \in B$. As $\widetilde{H}(0,0) = 0$, this implies $\widetilde{H}|_{\{0\}\times[0,1]}$ is the constant map to to $e \in \mathbb{R}$. The path $\widetilde{H}|_{[0,1]\times\{1\}}$ is a lift of δ, and we have just deduced this lift starts at $e \in E$, so it must be $\widetilde{\delta}$, as this was the unique such lift. A similar line of reasoning, but using the restrictions to $\{1\} \times [0,1]$, shows that $\widetilde{\gamma}(1) = \widetilde{\delta}(1)$. So the map ∂_e is well defined on homotopy classes of loops. □

Proposition 8.13. *If E is simply connected then the connecting map is a bijection.*

Proof. For surjectivity, let $e' \in p^{-1}(\{b\})$. As E is path connected, we may choose a path α in E from e to e'. Then $p \circ \alpha$ is a loop in B, based at b, and such that $\partial_e([p \circ \alpha]) = e'$. For injectivity, suppose that $\partial_e([\gamma]) = \partial_e([\gamma'])$. Then $\widetilde{\gamma}(1) = \widetilde{\gamma}'(1)$. As E is simply connected, there is a homotopy H (rel. e)from the loop $\widetilde{\gamma} \bullet (\widetilde{\gamma}')^{-1}$ to the constant loop at the basepoint e. Then $p \circ H$ is a homotopy (rel. b) from $\gamma \bullet \gamma')^{-1}$ to the constant loop at b. Thus $[\gamma] = [\gamma']$. □

*Proof that covering maps have the HLP**

Theorem 8.14. *Let $p\colon E \to B$ be a covering map. Then p has the homotopy lifting property.*

Proof. Fix Y, H, and h, as in the definition of the homotopy lifting property. We will prove existence and uniqueness of the lift "locally" near a point $y_0 \in Y$. We then say how to deduce the existence and uniqueness of the global lift from the local statement.

To show local existence, fix any $y_0 \in Y$. As f is continuous, for any $(y_0, t) \in Y \times I$ there exists an open set $N_t \times B(t; \varepsilon_t) \subseteq Y \times I$ around (y_0, t), such that $H(N_t \times B(t; \varepsilon_t)) \subseteq U_\alpha$ for some $\alpha \in I$. As $\{y_0\} \times I$ is compact, we can cover it with finitely many such $N_t \times B(t; \varepsilon_t)$. Write N for the intersection of the corresponding finite collection of N_t's (note that N is open). We can now choose a strictly increasing sequence

$$0 = t_0 < t_1 < t_2 < \cdots < t_m = 1$$

so that $H(N \times [t_i, t_{i+1}]) \subseteq U_\alpha$ for some $\alpha \in I$. Write U_i for this U_α. We now construct the lift $\widetilde{H}$ iteratively. Suppose we constructed the lift $\widetilde{H}$ on $N \times [0, t_i]$. To extend the lift to $N \times [t_i, t_{i+1}]$, consider that there is an open set $\widetilde{U}_i \subseteq \mathbb{R}$ that maps homeomorphically to U_i and is such that $\widetilde{H}(y_0, t_i) \in \widetilde{U}_i$. We may assume, after reducing the size of N, that $\widetilde{H}(N \times \{t_i\}) \subseteq \widetilde{U}_i$. Now define $\widetilde{H}$ on $N \times [t_i, t_{i+1}]$ to be the composition of $H\colon N \times [t_i, t_{i+1}] \to U_i$ followed by the homeomorphism $p^{-1}\colon U_i \to \widetilde{U}_i$. After finitely many iterations, there is a lift $\widetilde{H}$ on $N \times I$.

We now show the uniqueness of the local $\widetilde{H}$ just constructed. First, let $y \in Y$ be any point and assume we have a lift at $\{y\} \times I$. We will show this lift is unique. Suppose $\widetilde{H}$ and $\widetilde{H}'$ are two lifts at $\{y\} \times I$, such that $\widetilde{H}(y, 0) = \widetilde{H}'(y, 0)$. Using a decomposition into intervals $[t_i, t_{i+1}]$ as before, we see that, as p is a homeomorphism $p\colon \widetilde{U}_i \to U_i$, and the paths $\widetilde{H}$ and $\widetilde{H}'$ restricted to $\{y\} \times [t_i, t_{i+1}]$ both project to the same path under p, they must be the same path in $\widetilde{U}_i$. Starting at $i = 0$, and using the fact that $\widetilde{H}(y, 0) = \widetilde{H}'(y, 0)$, this uniqueness glues together along the intervals $\{y\} \times [t_i, t_{i+1}]$ as i increases. This shows $\widetilde{H}$ and $\widetilde{H}'$ agree on $\{y\} \times I$. But now this was independent of y, so the argument in fact shows the lift $\widetilde{H}$ on $N \times I$ constructed in the previous paragraph is unique.

Finally, we argue that the local version implies the global version. But this is clear – construct unique local lifts $\widetilde{H}$ on $N \times I$ at every $y_0 \in Y$. Where the lifts overlap, the uniqueness argument shows they agree on the overlaps. Thus they patch together to form a global, unique lift. □

The fundamental group of the circle and projective space

Theorem 8.15. *The fundamental group of the circle is $\pi_1(S^1, 1) \cong \mathbb{Z}$.*

Proof. The map

$$p\colon \mathbb{R} \to S^1; \qquad x \mapsto \exp(2\pi i x)$$

is a covering map, and therefore has the homotopy lifting property. Note that the fibre $p^{-1}(\{1\})$ is $\mathbb{Z} \subseteq \mathbb{R}$. As $\mathbb{R}$ is simply connected, Proposition 8.13 shows that the connecting map

$$\partial_0\colon \pi_1(S^1, 1) \to \mathbb{Z}$$

is a bijection. If we can show it is a group homomorphism, the theorem is proved.

To see that ∂_0 is a group homomorphism, let γ and δ be loops and consider that for any integer $N \in \mathbb{N}$, the path $\widetilde{\delta} + N$ is the unique lift of δ to a path starting at $N \in \mathbb{R}$. This shows that the (unique) lift of $\gamma \bullet \delta$ to $\mathbb{R}$ is the path $\widetilde{\gamma} \bullet (\widetilde{\delta} + \widetilde{\gamma}(1))$. This path has endpoint $\widetilde{\gamma}(1) + \widetilde{\delta}(1)$, so ∂_0 is a homomorphism. □

Theorem 8.16. *The fundamental group of real projective space* $\mathbb{R}P^n$ *is*

$$\pi_1(\mathbb{R}P^n; [\mathbf{x}]) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } n \geq 2 \\ \mathbb{Z} & \text{if } n = 1 \end{cases}$$

Proof. The quotient map

$$q\colon S^n \to \mathbb{R}P^n = S^n/\{\mathbf{x} \sim -\mathbf{x}\}; \qquad q(\mathbf{x}) = [\mathbf{x}]$$

is a covering map. To see this, fix $[\mathbf{x}] \in \mathbb{R}P^n$. Then we choose a small enough ball $B(\mathbf{x};\varepsilon) \subseteq \mathbb{R}^{n+1}$ so that for all $\mathbf{y} \in B(\mathbf{x};\varepsilon)$, we have $-\mathbf{y} \notin B(\mathbf{x};\varepsilon)$. Then $U_{\mathbf{x}} = q(S^n \cap B(\mathbf{x};\varepsilon))$ is an open set around $[x]$ with $q^{-1}(U_{\mathbf{x}}) = \widetilde{U}_+ \sqcup \widetilde{U}_-$, where $\widetilde{U}_\pm = S^n \cap B(\pm\mathbf{x};\varepsilon)$. Moreover $q\colon \widetilde{U}_\pm \to U$ is a homeomorphism.

For $n > 1$, we have that $\pi_1(S^n, e) = 0$, so that the connecting map $\partial_e\colon \pi_1(\mathbb{R}P^n, [\mathbf{x}]) \to \{\mathbf{x}, -\mathbf{x}\}$ is a bijection. As the only group of order 2 is $\mathbb{Z}/2\mathbb{Z}$, this proves the theorem.

If $n = 1$, we have that the map $f\colon S^1 \to S^1$ given by $f(z) = z^2$ is a continuous surjective map that descends to a bijection on the quotient $\mathbb{R}P^1 \to S^1$. As S^1 is compact and Hausdorff, this shows $\mathbb{R}P^1 \cong S^1$, and we have computed that $\pi_1(S^1, 1) \cong \mathbb{Z}$. □

9 *Applications of the fundamental group*

The Brouwer fixed-point theorem

We write

$$D^n = \{\mathbf{x} \in \mathbb{R}^n \mid ||\mathbf{x}|| \leq 1\}.$$

The disc D^2 is homeomorphic to a rectangle. Take two sheets of paper lying on top of each other on the table. Pick up the top sheet and scrunch it into a ball. Put it back down anywhere you like on the bottom sheet. By the Brouwer fixed-point theorem there is at least one point on the scrunched up ball sitting exactly above where it was when the sheet was flat.

Theorem 9.1 (The Brouwer fixed-point theorem)**.** *Any continuous map $f\colon D^2 \to D^2$ has a fixed point, i.e., a point $\mathbf{x} \in D^2$ such that $f(\mathbf{x}) = \mathbf{x}$.*

Proof. Suppose $f\colon D^2 \to D^2$ has no fixed point. Define $r\colon D^2 \to S^1$ to be the function that sends $\mathbf{x} \in D^2$ to the point on the circle where the ray that starts at $f(\mathbf{x})$ and passes through $\mathbf{x}$ intersects S^1. This is clearly a continuous map. Write $i\colon S^1 \to D^2$ for the inclusion map. We have a sequence of group homomorphisms

$$\underbrace{\pi_1(S^1, 1)}_{\cong \mathbb{Z}} \xrightarrow{i_*} \underbrace{\pi_1(D^2, 1)}_{=0} \xrightarrow{r_*} \underbrace{\pi_1(S^1, 1)}_{\cong \mathbb{Z}}.$$

But $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in S^1$ and thus this composition is the identity. This is a contradiction as there is no sequence of group isomorphisms $\mathbb{Z} \to 0 \to \mathbb{Z}$ that composes to be the identity map. □

The degree of a map

Definition 9.2. Let $f\colon S^1 \to S^1$ and let α be a path from $f(1)$ to 1. Fix an isomorphism $\pi_1(S^1, 1) \cong \mathbb{Z}$. Define an integer $\deg(f) \in \mathbb{Z}$, called the *degree* of f, to make the following diagram commute

$$\begin{array}{ccccc}
\pi_1(S^1,1) & \xrightarrow{f_*} & \pi_1(S^1, f(1)) & \underset{\cong}{\xrightarrow{C_\alpha}} & \pi_1(S^1,1) \\
\downarrow \cong & & & & \downarrow \cong \\
\mathbb{Z} & & \xrightarrow{\times \deg(f)} & & \mathbb{Z}.
\end{array}$$

Recall that

- Every self-homomorphism of the integers is multiplication by some integer n. We are denoting this map by $\times n\colon \mathbb{Z} \to \mathbb{Z}$.
- $C_\alpha([\gamma]) = [\alpha^{-1} \bullet \gamma \bullet \alpha]$).

(Some authors call $\deg(f)$ the *index* of f.)

Remark 9.3. If we think of S^1 as a subset of the complex plane, and $f\colon S^1 \to S^1$ as a loop, then the degree is equal to the *winding number*, which may be computed using a contour integral

$$\deg(f) = \frac{1}{2\pi i}\oint_f \frac{dz}{z},$$

as a special case of the Cauchy Integral Formula.

Proposition 9.4. *The degree of $f\colon S^1 \to S^1$ depends only on the homotopy class of f.*

Proof. First, a different choice of isomorphism $\pi_1(S^1, 1) \cong \mathbb{Z}$ would be multiplication by -1 on both vertical arrows in the diagram, so these overall cancel out and we get the same integer $\deg(f) \in \mathbb{Z}$. To see independence of α, recall that by Remark 7.13 a different choice would change the map in the top line overall by a conjugation. But as $\mathbb{Z}$ is abelian this is the identity map. Finally, by Proposition 7.24, the integer depends only on the homotopy class of f. □

We will now present a couple of applications of the degree.

Theorem 9.5 (The fundamental theorem of algebra)**.** *For some $n > 0$, let*

$$p(z) = z^n + c_{n-1}z^{n-1} + \cdots + c_1 z + c_0$$

be a polynomial with complex coefficients. Then there exists $z \in \mathbb{C}$ such that $p(z) = 0$.

Proof. Suppose that p has no zeros. Then the following is well-defined:

$$\widehat{p}\colon S^1 \to S^1; \qquad z \mapsto \frac{p(z)}{||p(z)||}.$$

We show that the degree of this map is both n and 0, which is a contradication, as $n > 0$.
As p has no zeros, the homotopy $H(z,t) = p(tz)/||p(tz)||$ is well-defined. At $t = 0$, this is the constant map

$$S^1 \to S^1; \qquad x \mapsto f(0)/||f(0)||,$$

and at $t = 1$, this is $\widehat{p}$. But the constant map induces the 0-map on homotopy groups, so this implies $\deg(\widehat{p}) = 0$.
Consider as well that, as p has no zeros, there is a homotopy $J(z,t) = t^n p(z/t)$ from the function $z \mapsto z^n$ to p. Thus $J(z,t)/||J(z,t)||$ is a homotopy from $z \mapsto z^n$ to $\widehat{p}$. As the former clearly has degree n, this shows $\deg(\widehat{p}) = n$. □

The Borsuk-Ulam theorem

Theorem 9.6 (The Borsuk-Ulam Theorem)**.** *Let $f\colon S^2 \to \mathbb{R}^2$ be a map. Then there exist antipodal points $\mathbf{x}$ and $-\mathbf{x}$ on S^2, such that $f(-\mathbf{x}) = -f(\mathbf{x})$.*

For example, let f represent the NSEW direction the wind is blowing at a point on the Earth. The theorem says at any given time you can find antipodal points on the earth where the wind is blowing exactly in the opposite cardinal direction.

Proof. The theorem is equivalent to showing that the map

$$g\colon S^2 \to \mathbb{R}^2; \qquad g(\mathbf{x}) = f(\mathbf{x}) - f(-\mathbf{x})$$

must be 0 for some $\mathbf{x}$. Note that $g(\mathbf{x}) = -g(\mathbf{x})$. Consider projecting the unit disc in the complex plane onto the upper hemisphere of S^2 via the map

$$D^2 \to S^2; \qquad (x+iy) \mapsto (x, y, \sqrt{1-x^2-y^2}).$$

From this we obtain a map

$$h\colon D^2 \to \mathbb{R}^2; \qquad h(x+iy) = g(x, y, \sqrt{1-x^2-y^2}).$$

Observe that the map h inherits from g the property that $h(z) = -h(z)$. If we can show that h vanishes at some point z on D^2, we can project it to the upper hemisphere of S^2 and we have found a point on S^2 where g vanishes. We now show such a vanishing point for h exists.

Suppose there does not exist $z \in D^2$ such that $h(z) = 0$. Then we may define a continuous function

$$\varphi\colon D^2 \to S^1; \qquad \varphi(z) = \frac{h(z)}{||h(z)||}\frac{||h(1)||}{h(1)}.$$

Note that $\varphi(z) = -\varphi(-z)$ and $\varphi(1) = 1$. The map

$$\gamma\colon [0,1] \to D^2; \qquad \gamma(t) = \exp(2\pi i t)$$

is homotopic to the constant loop c_1 via some homotopy H. Postcomposing with φ, we see that the loop $\varphi \circ \gamma$ in S^1 is homotopic to the constant map via the homotopy $\varphi \circ H$. Thus $\varphi \circ \gamma$ has degree 0. We will now argue that the $\deg(\varphi \circ \gamma)$ is odd, giving a contradiction.

Let $k\colon [0,1] \to \mathbb{R}$ be the unique lift along the covering map $\mathbb{R} \to S^1$ of the loop $\varphi \circ \gamma$, in other words $\exp(2\pi i k(s)) = \varphi(\exp(2\pi i s))$. By definition of the degree, we have $\deg(\varphi \circ \gamma) = k(1)$. Consider that as $\varphi(z) = -\varphi(-z)$, we have

$$\begin{aligned}\exp(2\pi i k(s)) = \varphi(\exp(2\pi i s)) &= -\varphi(-\exp(2\pi i s)) \\ &= -\varphi(\exp(2\pi i(s+\tfrac{1}{2})) = -\exp(2\pi i k(s+\tfrac{1}{2})).\end{aligned}$$

Thus

$$\exp(2\pi i k(s)) = -\exp(2\pi i k(s+\tfrac{1}{2})) = \exp(2\pi i(k(s+\tfrac{1}{2}) - \tfrac{1}{2}),$$

so

$$(k(s+\tfrac{1}{2}) + \tfrac{1}{2}) - k(s) \in \mathbb{Z}$$

for every $s \in [0, 1/2]$. But this is a continuous function of s, and hence is a constant because $\mathbb{Z}$ is discrete. So there exists $m \in \mathbb{Z}$ such that

$$k(s + \tfrac{1}{2}) - k(s) = m + \tfrac{1}{2}.$$

We compute

$$\begin{aligned} k(1) &= (k(1) - k(1/2)) + (k(1/2) - k(0)) \\ &= (m + \tfrac{1}{2}) + (m + \tfrac{1}{2}) \\ &= 2m + 1 \end{aligned}$$

is odd. □