Lecture 1

# §I Introduction

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## Standing notation

X - finite CW cx

M - n-dim closed CAT manifold where CAT = Diff, PL, TOP

corresponds to alg surgery.

**Def'n** $\mathcal{S}^{cat}(X) := \{ f : M^n \xrightarrow{\simeq} X^n \} / \sim$

where $f_0 \sim f_1$ iff $\exists$ iso $h : M_0 \xrightarrow{\cong} M_1$ s.t.

$$\begin{tikzcd} M_0 \arrow[dr, "f_0"] \arrow[d, "h"'] & \\ M_1 \arrow[r, "f_1"'] & X \end{tikzcd}$$

htpy commutes

CAT structure set.

**Q1** When is $\mathcal{S}(X)$ non-empty?

**Q2** How do we calculate $\mathcal{S}(X)$? if it is non-empty.

RE: Q1, lets look at 3 properties of manifolds.

(T) $\exists$ tangent bundles $\mathbb{R}^n \to TM \to M$

(ω) $\pi = \pi_1(M) \xrightarrow{\omega} \mathbb{Z}_2$ the orientation character

M orientable $\Leftrightarrow \omega = 0$

(PD) Poincaré duality. $\exists [M] \in H_n(M; \mathbb{Z}^\omega)$ s.t.

$$- \cap [M] : H^i(M; \mathbb{Z}\pi) \xrightarrow{\cong} H_{n-i}(M; \mathbb{Z}\pi)$$

**Def'n** A Poincaré cx is a triple $(X, \omega, [X])$ where

- $\omega : \pi_1(X) \longrightarrow \mathbb{Z}_2$
- X a finite CW cx
- $[X]$ is a fundamental class determining PD.

**Remark** In fact, ω is determined by X.

<u>Observation</u> If $\mathcal{S}^{cat}(X) \neq \phi$ then $X$ is a Poincaré cx.

<u>Theorem</u> Every PC $X = (X, \boxed{\cancel{[X]}})$ admits a SNF $\nu_X$ (Spiv. normal fib). This is a model for the normal bundle of $M^n \hookrightarrow \mathbb{R}^{n+k}$ (L2). s.t. if $X \simeq M$ then $\nu_X$ has a bundle reduction $\xi$. $\square$

We also have the collapse map for f

$$S^{n+k} \xrightarrow{c} Th(\nu_M) = D(\nu_M)/S(\nu_M)$$

Moreover, the SNF has htpy. class

$$c: S^{n+k} \longrightarrow Th(\nu_X)$$

If $\nu_X$ has a reduction $\xi$. Consider

$$\begin{tikzcd} & BO \arrow[d] \\ X \arrow[ur, "\xi"] \arrow[r, "\nu_X"] & BG \end{tikzcd}$$

$$c: S^{n+k} \longrightarrow Th(\nu_X) \longrightarrow Th(\xi)$$

$X \subset Th(\xi) \Rightarrow$ by transversality we take $c^{-1}(X) =: M$

$$\Rightarrow \left\{ \begin{tikzcd} \nu_M \arrow[r, "\overline{F}"] \arrow[d] & \xi \arrow[d] \\ M \arrow[r, "F"] & X \end{tikzcd} \right\} \Big/ \text{normal bordism}$$

A degree 1 normal map

$$=: \mathcal{N}(X)$$

Surgery gives a map

$$\sigma : \mathcal{N}(X) \longrightarrow L_n(\mathbb{Z}\pi)$$

s.t. $\sigma = 0 \underset{(n\geq 5)}{\overset{\Longleftarrow}{\Longrightarrow}} (M, f, \bar{F}) \sim (M', f', \bar{F}')$ with $f'$ a htpy equiv

Thus $\mathcal{S}^{CAT}(X)$ is non-empty $\Leftrightarrow \exists (\xi, c)$ s.t.

$$\sigma(\bar{F}_{\xi,c}, f_{\xi,c}) = 0 \in L_n(\mathbb{Z}\pi)$$

If $\mathcal{S}^{CAT}(X)$ is non-empty, we consider the uniqueness question.

<u>Th'm</u> (B-N-S-W) $\exists$ a LES of pointed sets (CAT fixed)

$$\cdots \to L_{n+1}(\mathbb{Z}\pi) \longrightarrow \mathcal{S}(X) \longrightarrow \mathcal{N}(X) \xrightarrow{\sigma} L_n(\mathbb{Z}\pi)$$

Shmuel-Weinberger "Uniqueness is a relative version of existence"

<u>Th'm</u> This sequence has an algebraic twin when CAT = TOP

$$\cdots \to L_{n+1}(\mathbb{Z}\pi) \longrightarrow \$_{n+1}(X) \longrightarrow H_n(X; \mathbb{L}_\bullet\langle 1\rangle) \longrightarrow L_n(\mathbb{Z}\pi)$$

and groups are isomorphic.

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~~§2 Bundles~~ With this technology we have

<u>Th'm</u> (Browder/Novikov) If $X$ is 1-conn and SNF lifts to a PL bundle $\Rightarrow \mathcal{S}^{PL}(X) \neq \emptyset$.

## §2 Bundles

$B$ = base space finite CW cx

Recall $\mathrm{Vect}_k(B) := \{ [E] \mid E \to B \ \ \mathrm{rk} = k \}$

Assume $E$ has a metric

Th'm $\exists$ a universal vector bundle

$$VO(k) \longrightarrow BO(k)$$

with a bijection $\mathrm{Vect}_k(B) = [B, BO(k)]$.

NOTE the structure group is $GL(k)$. We want to weaken this to $\mathrm{Homeo}(\mathbb{R}^k, \mathbb{R}^k)$.

E.g. • $B \times \mathbb{R}^k =: \underline{\mathbb{R}}^k$

• $f: M^n \looparrowright V^{n+k}$ ~~if~~ $f$ an immersion

$$\Rightarrow f^* TV \cong TM \oplus \nu f$$

• $M^n \hookrightarrow \mathbb{R}^{n+k}$, $k \gg n$ then $\nu_f$ is the stable normal bundle.

Def'n A stable vector bundle over $X_i$ is a sequence $\{E_i, \alpha_i\}$ where

$$\begin{array}{ccc} \alpha_i : E_{i+1} & \xleftarrow{\cong} & E_i \oplus \underline{\mathbb{R}} \\ \downarrow & & \downarrow \\ X_{i+1} & \longleftarrow & X_i \end{array}$$

E.g. $M^n \hookrightarrow \mathbb{R}^{n+k} \subseteq \mathbb{R}^{n+k+1} \subseteq \mathbb{R}^{n+k+2} \subseteq \ldots$

$\Rightarrow \nu_{f_k} \oplus \underline{\mathbb{R}} \cong \nu_{f_{k+1}}$

A stable vector bundle gives rise to a Thom spectrum

$$M(\xi) := \{ Th(E_i), \text{ obvious maps} \}$$

Recall $Th(E_i) = D(E_i)/S(E_i)$ and $Th(E_i \oplus \underline{\mathbb{R}}) \cong \Sigma Th(E_i)$

i.e. a SPECTRUM

## The Thom-Spectrum

E.g. $MO := M(\xi)$ where $\xi = \{VO(k)\}$ and $X_i = BO(k)$

Th'm (Pont-Thom) $\exists$ an isomorphism

$$\pi_n(M(\xi)) \cong \Omega_n(\xi)$$

stable htpy groups of $M(\xi)$

bordism of manifolds with $\xi$-structure

where $\Omega_n(\xi)$ consists of equivalence classes

$$\begin{array}{ccccc} & & \nu_M & \xrightarrow{\bar{f}_k} & \xi_k \\ & & \downarrow & & \downarrow \\ S^{n+k} & \hookleftarrow & M & \xrightarrow{f_k} & X_k \end{array}$$

IDEA OF PROOF apply transversality to a map $S^{n+k} \xrightarrow{c} Th(E(\xi_k))$ to pull back $X_k$ and get a manifold $[M] \in \Omega_n(\xi)$. with $\nu_M \cong c|_M^* \xi_k$.

## §3 Microbundles

Def'n A microbundle/$\xi$ (rank $k$) is a triple $(E, B, i, j)$ where ($E$, $B$ spaces)

$$B \xrightarrow{i} E \xrightarrow{j} B$$

s.t.
(i) $j \circ i = id$
(ii) $\forall b \in B \; \exists \; U \ni b$ and $V \ni i(b)$ with $i(U) \subseteq V$ and $j(V) \subseteq U$.

and we assume a homeo $h : V \xrightarrow{\cong} U \times \mathbb{R}^k$ s.t.

$$\begin{tikzcd} & V \arrow[dr, "j|_V"] \arrow[d] & \\ U \arrow[ur, "i|_U"] \arrow[r, "\times 0"'] & U \times \mathbb{R}^n \arrow[r, "p_1"'] & U \end{tikzcd}$$ commutes

<u>E.g.</u> Define when $(E_1, B, i_1, j_1) \cong (E_2, B, i_2, j_2)$

$\exists$ n'hoods $V_a \supseteq i_a(B)$ $a = 0$ or $1$.
and a homeo

$$h : V_1 \xrightarrow{\cong} V_2$$

s.t.

$$\begin{tikzcd} & V_1 \arrow[dr, "j_1|V_2"] \arrow[d, "h"] & \\ B \arrow[ur, "i_1"] \arrow[r, "i_2"'] & V_2 \arrow[r, "j_2|V_2"'] & B \end{tikzcd}$$ commutes.

<u>E.g.</u> $M$ a topological manifold, then form

$$\xi_M = (M, M \times M, \Delta_M, pr_1.)$$

EXERCISE - This is a microbundle.

<u>E.g.</u> If $\mathbb{R}^k \to E \xrightarrow{\pi} B$ is a linear v.b. then

$(E, B, s, \pi)$ is a microbundle for $s$ the zero section.

<u>Th'm</u> (Milnor) If $M$ is smooth then $TM \cong \xi_M$ as microbundles.

Lecture 2

Diarmuid Crowley

## Topological bundles cont.

**Recall** The frame bundle of v.b. $E^k$

$$FE \subseteq \underbrace{E \times \dots \times E}_{k \text{ times}}$$
$$= \{(v_1, \dots, v_k) \mid \text{this is an o.n. basis of some } E_x\}$$

Observe $\exists$ a cont action of $O(k)$ on $FE$ and a principal $O(k)$-bundle

$$O(k) \to FE \to X$$

and $E \cong FE \times_{O(k)} \mathbb{R}^k$.

For any topological group $G$, $\exists$ a universal bundle

$$G \to EG \to BG \quad \text{with } EG \simeq pt.$$

If $G$ acts on a space $F$ we obtain the universal bundle

$$F \to EG \times_G F \to BG$$

of $F$-bundles with structure group $G$.

E.g. for $G = \mathrm{Homeo}_*(\mathbb{R}^k)$ base point-preserving homeos $\mathbb{R}^k \to \mathbb{R}^k$
$=: TOP_k$

$\Rightarrow \exists$ spaces $ETOP_k \to BTOP_k$ and

$$\mathbb{R}^k \to TOP_k \to BTOP_k$$

Observe $TOP_k \to TOP_{k+1}$ by $f \mapsto f \times id_{\mathbb{R}}$

$\Rightarrow$ $\mathrm{TOP} := \lim_{k\to\infty} \mathrm{TOP}_k$

the notion of stable bundles
and

$$\mathrm{BTOP} := \lim_{k\to\infty} \mathrm{BTOP}_k$$

$\pi_i(\mathrm{TOP}_k)$ ? Reduces to computing htpy groups of spheres.

**Th'm** Kister, Lashof + Rosenberg, Mazur

Let $B \xrightarrow{i} E \xrightarrow{j} B$ be a $\mathrm{rk}=k$ microbundle. Then $\exists\, E_i \subset E$, $E_i \supset i(B)$ s.t.
- $E_i = E_i(\xi)$ is the total space of a topological $\mathbb{R}^k$-bundle over $B$.
- The inclusion $E_i \hookrightarrow E$ is an isomorphism of microbundles
- $E_i$ is uniquely determined up to $\mathbb{R}^k$-bundle iso

"Microbundles are bundles".

**Cor** Every TOP manifold $M^n$ has

$$M\times M \supseteq TM \longrightarrow M \longrightarrow \mathrm{BTOP}_n$$

an honest $\mathbb{R}^n$-bundle, its tangent bundle.

**Theorem** Transversality holds for $\mathrm{TOP}_n$ bundles.

$n>4$ (Kirby-Siebenman)
$n=4$ (Freedman-Quinn)

i.e. Given $c: M^{n+k} \longrightarrow E(\xi)$
s.t. $c|_{\partial M} \pitchfork X$

where $M^{n+k}$ is a TOP-mfld
$\xi$ is an $\mathbb{R}^k$-bundle over $X$

which is to say $\partial M$ is a codim 1 mfld that is locally flatly embedded.

we should get some $c' \underset{\mathrm{rel}\ \partial M}{\simeq} c$ st. $c' \pitchfork X$

**Moreover** $N := (c')^{-1}X$ has an $\mathbb{R}^k$-normal bundle iso to $(c')^{*}\xi$

[Recall "locally flat" means $\forall x \in N\ \exists\, U \ni x$ and $V \ni f(x)$ and

$$\begin{array}{ccc} U & \xrightarrow{\cong} & \mathbb{R}^n \\ \downarrow i' & & \downarrow \times 0 \\ X & \xrightarrow{\cong} & \mathbb{R}^{n+k} \end{array}$$
]

__E.g.__ Sanderson '66 $\exists$ topological embeddings without normal bundle.

__Corr__ $\exists$ a Pont.-Thom in TOP

$$\Omega_n(\xi) \cong \pi_n(Th(\xi))$$

where $\xi$ a stable topological bundle.

__Remark__ If $\xi$ has a v.b. reduction then $\xi'$

$$\Omega_n^{TOP}(\xi) \cong \pi_n(Th(\xi)) \cong \pi_n(Th(\xi')) \cong \Omega_n^{Diff}(\xi').$$

__Def'n__ If $\mathbb{R}^k \to E \to X$ is a topological $\mathbb{R}^k$-bundle then

$$Th(E) := E \cup \{\infty\} \quad \text{if } X \text{ is compact.}$$

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__Spherical Fibrations__

__Def'n__ A spherical fibration of rank $k$ $\xi$ is a fibration

homotopy fibre

$$S^{k-1} \longrightarrow S(\xi) \xrightarrow{\pi} X$$

The disk bundle of $\xi$ is $D(\xi) := cyl(\pi)$ and this is a fibration

$$D^k \longrightarrow D(\xi) \longrightarrow X$$

E.g. $\xi$ a v.b. then we take a metric and form $S(\xi), D(\xi)$

$$S(\xi) := E(\xi) \setminus s_0(X)$$

zero section

Def'n A fibre htpy equivalence $\xi_0 \simeq \xi_1 \iff \exists\, f$

$$\begin{tikzcd} S(\xi_0) \arrow[r, "f"] \arrow[d] & S(\xi_1) \arrow[d] \\ X \arrow[r, equal] & X \end{tikzcd}$$

commutes.

Def'n $SF_k(X) := \{ [E(\xi)] \mid \xi \text{ rank } k \text{ spherical} \}$

E.g. • $X \times S^{k-1} \longrightarrow X$

Fibrewise join with $X \times S^0$ defines stabilisation

Def'n A stable spherical fibration $\{\xi_i, \alpha_i\}$

$$\alpha_i : E(\xi_i *_X S^0) \longrightarrow E(\xi_{i+1})$$

$$Th(\xi) := D(\xi)/S(\xi)$$

[Remark: Can use this to define Thom of an $\mathbb{R}^k$-bundle]

We have forgetful maps

$$VB_k(X) \longrightarrow VTOP_k(X) \longrightarrow SF_k(X)$$

stabilise $\quad SVB(X) \longrightarrow SVTOP(X) \longrightarrow SSF(X)$

**Remark** Spherical fibrations are not bundles in general so do not have a structure group. But we do have a structure monoid

**Def'n** $G(k) := Map_{\pm}(S^{k-1}, S^{k-1})$ with composition.

$$G := \lim_{k\to\infty} G(k)$$

$\exists$ forgetful maps

$$O(k) \longrightarrow TOP(k) \xrightarrow{\text{pt compactify}} G(k+1)$$

$$O \longrightarrow TOP \longrightarrow G$$

Given $E(\xi)$ a spherical fib of $rk = k$ define

$$P(\xi) \subset Map_{\bullet}(S^{k-1}, E(\xi))$$
$$\| $$
$$\{ f : S^{k-1} \xrightarrow{\simeq} E(\xi)_x \}$$

Observe that $G(k)$ acts, by precomposition, on $P(\xi)$

$$\Rightarrow \quad G(k) \to P(\xi) \to X$$

$\boxed{\text{FACT}}$ Topological monoids have classifying spaces

Milgram
Steenrod

**Th'm** (Stasheff) $\exists$ a universal spherical fibration $VG(k) \to BG(k)$ and a bijection

$$SF_k(X) \equiv [X, BG(k)]$$

and

$$SSF(X) \equiv [X, BG]$$

**Lemma** (Milgram-Madsen book) $\exists$ a fibration sequence

$$F_k \longrightarrow G(k+1) \xrightarrow{ev} S^k$$
$$f \longmapsto f(x_0)$$

($F_k$ = basepoint preserving $f: S^k \to S^k$ htpy equiv)

and we can show $F_k \underbrace{\simeq}_{\text{use adjunction}} \Omega^k_{+1} S^k \cup \Omega^k_{-1} S^k$

**Corr** $\pi_i(G(k)) \cong \pi_i(F_k) \cong \pi_{k+i}(S^k) \cong \pi_i^s$ (for $i \leq k-2$)

**Th'm** The induced map

$$J_i : \pi_i(O) \longrightarrow \pi_i(G) \cong \pi_i^s$$

is the (stable) J-homomorphism.

**Th'm** (Bott) $\pi_i(O) = \mathbb{Z}_2\ \mathbb{Z}_2\ 0\ \mathbb{Z}\ 0\ 0\ 0\ \mathbb{Z}$

**Th'm** (Adams)
- $J_i : \pi_i(O) \longrightarrow \pi_i(G)$ injective for $i = 0, 1$.
- $J_{4k-1}(\mathbb{Z})$ is a cyclic summand of order denom$\left(\frac{B_k}{4k}\right)$

**Th'm** (Serre) $|\pi_i(G)| < \infty$ if $i > 0$.

**Th'm** (Kerv-Miln) $\exists$ a SES $\pi_i(O) \longrightarrow \pi_i(PL) \longrightarrow \pi_i(PL/O)$

where $\pi_i(PL/O)$ is finite.

**Th'm** (Cerf) $\pi_i(PL/O)$ is 6-connected

**Th'm** (Kirby-Siebenmann) $TOP/PL \simeq K(\mathbb{Z}_2, 3)$

SO!

$$\pi_i(TOP) \to \pi_i(G) \to \pi_i(G/TOP)$$

finite

$\Rightarrow \otimes \mathbb{Q}$ give $\pi_i(TOP) \otimes \mathbb{Q} \cong \pi_{i+1}(G/TOP) \otimes \mathbb{Q}$

$\pi_i(O) \otimes \mathbb{Q} \cong$

In fact, surgery will show us that

$$\pi_i(G/TOP) \cong L_i(\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0 \\ 0 & i=1 \\ \mathbb{Z}_2 & i=2 \\ 0 & i=3 \end{cases}$$

**Th'm** (Bordman Vogt) There exist $\infty$-loop space structures on

$$O \to PL \to TOP \to G \quad \text{and } \infty\text{-loop maps.}$$

$\Rightarrow G/CAT$, $TOP/CAT$ and compatible $\infty$-loop spaces.

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Two explicit definitions of $J_i : \pi_i(O) \to \pi_i(G)$.

(1) $\alpha \in \pi_i(O(k+1)) \Rightarrow f_\alpha : S^i \to \mathrm{Map}_\pm(S^k, S^k)$

"Adjoin": $F_\alpha : S^i \times S^k \longrightarrow S^k$

$(x, y) \longmapsto f_\alpha(x)(y)$

$\Rightarrow \Sigma F_\alpha \quad S^i * S^k \longrightarrow \Sigma^k$

$S^{i+k+1} \longrightarrow S^{k+1}$

$\Rightarrow J(\alpha) = [\Sigma F_\alpha]$

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(2) $k \geq i+2 \qquad \pi_i(O) \to \pi_i(G) \cong \pi_i^s \cong \Omega_i^{fr}$

$\alpha \in \pi_i(O(k))$, take $S^i \times D^k \hookrightarrow S^{i+k}$ with standard framing.

$J(\alpha) := [(S^i, F_\alpha)]$ exotic framing (twisted by $\alpha$)

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<u>Lemma</u> (Milnor) Let $\xi_\alpha \to S^i$ be a rk $= k$ vector bundle. $\alpha \in \pi_{i-1}(SO(k))$ Then

$$Th(\xi_\alpha) \simeq S^k \cup_\varphi e^{i+k}$$

where $\varphi : S^{i+k-1} \longrightarrow S^k$ where $\varphi = J(\alpha)$

This defines $\pi_{i-1}(SO(k)) \xrightarrow{J} \pi_{i+k-1}(S^k)$ (unstable J-homo)

# Surgery below the middle dimension

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## Proof of highly connectedness (cont.)

Idea: inductively kill $\pi_{k+1}(f)$.

Let $\alpha \in \pi_{k+1}(f) \Rightarrow$

$$\begin{array}{ccc} D^{n-k} \times S^k & \xrightarrow{\alpha} & M \\ \downarrow & & \downarrow \\ D^{n-k} \times D^{k+1} & \xrightarrow{\alpha'} & X \end{array}$$

covered by ~~normal~~ tangent data.

Now note $(\alpha')^* \xi = \underline{\mathbb{R}}^j \Rightarrow T(D^{n-k} \times D^{k+1}) = \underline{\mathbb{R}}^n$

**Theorem** (Hirsch-Smale) Let $M^m$ and $N^n$ be smooth mflds with $N$ closed. Assume $M$ has a handle decomposition with $q$-handles for $q \leq n-2$ and $1 \leq m \leq n$. Then taking the differential gives a bijection

i.e. regular htpy classes of immersions

htpy classes of bundle monomorphisms

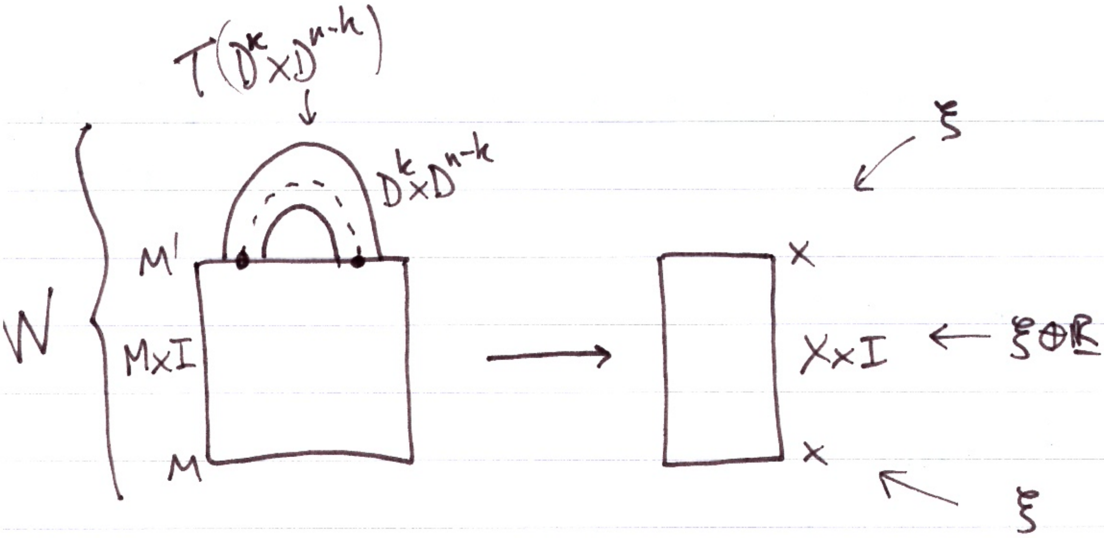
$$\pi_0 \operatorname{Imm}(M, N) \underset{\substack{\text{colim} \\ a \to \infty}}{\equiv} \pi_0 \operatorname{Mono}(TM \oplus \underline{\mathbb{R}}^a, TN \oplus \underline{\mathbb{R}}^a)$$

$$[f : M \looparrowright N] \qquad\qquad [Tf \oplus \mathrm{id}_{\underline{\mathbb{R}}^a}]$$

$\square$

**Remark** (i) This is a version of the h-principle.
(ii) At least if $m = n$ a similar statement holds in TOP.

Now for $k < q$ we can assume that $\alpha : S^k \to M$ is an embedding and the bundle data ensure that $\alpha$ has a trivial normal bundle.




$W :=$ trace of surgery on $\alpha$

$W \simeq M \cup e^k \simeq M' \cup e^{n-k}$

$\Rightarrow \pi_k(M') \cong \pi_k(W) \cong \pi_k(M)/\langle\alpha\rangle$ $\quad$ $\mathbb{Z}\pi$-module generated by $\alpha$.

$\boxed{\text{EXERCISE}}$ $\quad \pi_{k+1}(f)$ always f.g. over $\mathbb{Z}\pi$ $\quad$ (in partic this process of killing terminates)

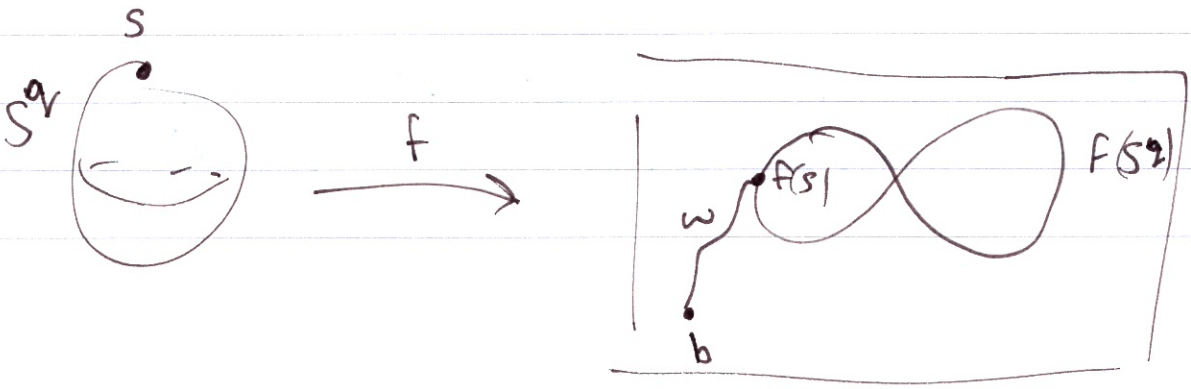
$\quad \cong$

$H_{k+1}(\tilde{X}, \tilde{M}; \mathbb{Z})$ regard $\tilde{M} \hookrightarrow \tilde{X}$ $\quad \square$

## Surgery in the middle dimension

**Def'n** $I_q(M) := \{(f, \omega)\} =$ reg htpy classes of immersions $f$ and paths $\omega$ from basepoint to $f(s)$ (where we have fixed $s \in S^q$).

A group.



$I^\circ_q(M) \subseteq I_k(M)$, immersions with trivial normal bundle.

$\boxed{\text{EXERCISE}}$ (Hirsch-Smale) $I^\circ_q(M) \xrightarrow{\cong} \pi_q(M)$ as groups.

## Intersections

**Lemma** There exists a sesquilinear form $\lambda$

$$\begin{tikzcd} I^\circ_q(M) \times I^\circ_q(M) \arrow[r, "\lambda"] \arrow[d] & \mathbb{Z}\pi \arrow[d, equal] \\ H_q(\tilde{M}) \times H_q(\tilde{M}) \arrow[r, "\lambda_{\tilde{M}}"] & \mathbb{Z}\pi \end{tikzcd}$$ commutes

**Remark** $I^\circ_k(M)$ is a $\mathbb{Z}\pi$-module; $\gamma \in \pi_1(M,b)$ then the action $(f,w) \longmapsto (f, w\circ\gamma)$.

**Proof/construction** $f_0 \neq f_1 \in I^\circ_q(M)$ then make $f_0 \pitchfork f_1$ (within reg htpy class)

$\Rightarrow \operatorname{Im}(f_0) \cap \operatorname{Im}(f_1) =$ finite set of points $=: D$.

For $x \in D \rightsquigarrow \varepsilon(x) g(x)$ where $g \in \pi_1(M,b)$ ~~brings~~

$\varepsilon(x)$ : compare signs of tangent spaces using paths in $S^q$ from $s$ to $f_i^{-1}(x)$

for $g(x)$

Use the chosen paths in $S^q$ and $w_0, w_1$ to obtain a loop $\gamma$ based at $b \Rightarrow g := [\gamma]$.

$$\lambda(f_0, f_1) := \sum_{x \in D} \varepsilon(x)\, g(x).$$

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For $f_0 = f_1 =: f$ we need to be more careful. There is now no clear choice of which $f_i$ should go first. The ambiguity from the ordering of the process lands us in $\mathbb{Z}\pi / x - \varepsilon\bar{x} = \hat{Q}^\varepsilon(\mathbb{Z}\pi)$

$\varepsilon = (-1)^q$

$$\mu(f) = \sum_D$$

$\bar{x} = \sum_g w(g) g^{-1}$

the w-twisted inv.

**Prop** $(I_k^\circ(M), \lambda, \mu)$ is a $\varepsilon$-quadratic form

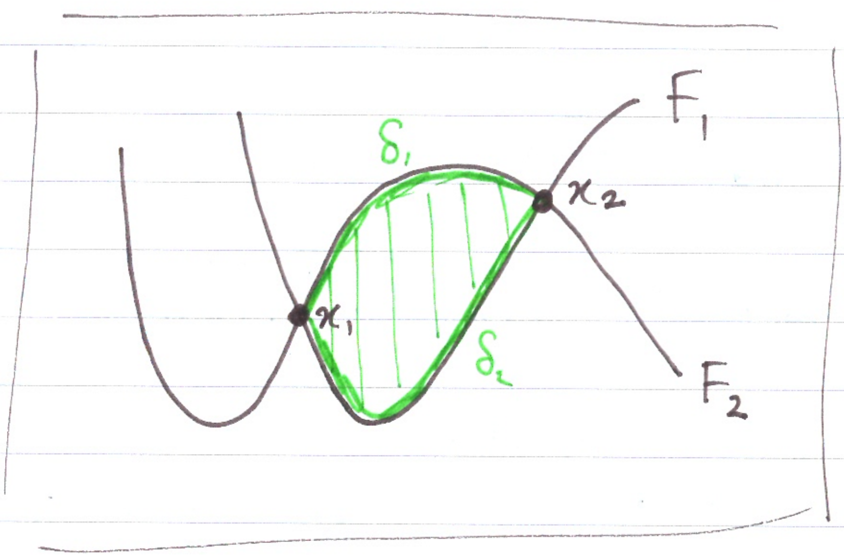
- $\lambda(\alpha, \beta) = \varepsilon \overline{\lambda(\beta, \alpha)}$
- $\lambda(\alpha, u_1\beta_1 + u_2\beta_2) = u_1 \lambda(\alpha, \beta_1) + u_2 \lambda(\alpha, \beta_2)$
- $\mu(\alpha + \beta) = \mu(\alpha) + \mu(\beta) + [\lambda(\alpha, \beta)]$
- $\lambda(\alpha, \alpha) = (1 + T_\varepsilon)\mu(\alpha) \in \mathbb{Z}\pi$.
- $\mu(r\alpha) = r\mu(\alpha)\bar{r}$ $\square$

**Theorem** (Wall)
- $q \geq 3$ then $\mu(\alpha) = 0 \Leftrightarrow \exists [f] = \alpha \in I_q^\circ(M)$ with $f$ an embedding.
- $\lambda(\alpha, \beta) = 0 \Leftrightarrow$ can represent disjointly

**Proof** (idea) Whitney trick

Local picture

$M^{2q}$




$F_i \subseteq \mathrm{Im}(f_i)$

$x_1, x_2$ pair up so that $\varepsilon(x_1) = -\varepsilon(x_2)$ and $g(x_1) = g(x_2)^{-1}$

$\Rightarrow \delta_1 \circ \delta_2 = 1$

$\Rightarrow$ bounds <u>embedded</u> disk $D^2$

$q \geq 3$

So locally we have $D^2 \times D^{2q-2}$ and can show it is $\left\{\begin{array}{l}\text{diffeo}\\ \text{homeo}\end{array}\right.$ to

$$D^2 \times D^{2q-2} \supseteq D^2 \times \Big(\underset{F_1}{\underset{\cap}{D^{q-1}}} \times \underset{F_2}{\underset{\cap}{D^{q-1}}}\Big)$$

Now do a Whitney move. $\square$

## <u>Even dimensional surgery obstruction.</u>

<u>Def'n</u> $K_q(M) := \pi_{q+1}(f)$

$\boxed{\text{EXERCISE}}$ This is a stably f.g. free $\mathbb{Z}\pi$-module

$\Rightarrow M \#_r (S^q \times S^q) \longrightarrow X$ is normally bordant

$\rightarrow$ Assume $K_q(M)$ is even free over $\mathbb{Z}\pi$.

Set $\sigma(\bar{f}, f) = [(K_q(M), \lambda, \mu)] \in L_{2q}(\mathbb{Z}\pi)$

Where $K_q(M) \xrightarrow{\partial} \pi_q(M) \cong I^{\circ}_q(M)$ gives $\lambda, \mu$.

<u>Def'n</u> $L_{2q}(\mathbb{Z}\pi) := \left\{ (P, \lambda, \mu) \;\middle|\; \begin{array}{l} P \text{ f.g. free} \\ \lambda : P \to P^* \end{array} \quad \lambda, \mu \text{ quadratic} \right\} \Big/ \sim$

- $\sim$ generated by isometry
- Metabolics $= 0$ (lemma: $\Leftrightarrow$ hyperbolics $= 0$)

<u>Def'n</u> • $L \subseteq P$ Lagrangian if
(i) Half rank summand
(ii) $\lambda|_{L \times L} \equiv \mu|_L \equiv 0$
• Metabolic $\Leftrightarrow$ has a Lagrangian

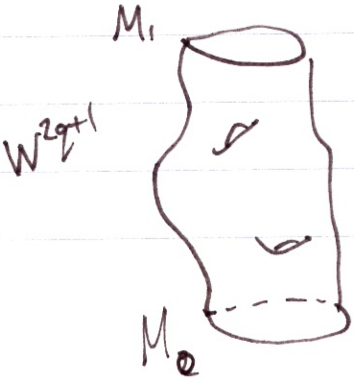
<u>Lemma</u> $(P, \lambda, \mu) \supseteq (L, 0, 0)$ extends to an iso

$$H_\varepsilon(L) \cong (P, \lambda, \mu)$$

where $H_\varepsilon(L) = \left(L \oplus L^*, \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}, \mu(L) = 0 = \mu(L^*)\right)$

<u>Prop</u> $\sigma(\bar{F}, f) = 0 \Leftrightarrow (\bar{F}, f)$ normally bordant to a htpy equiv

<u>Proof</u> Show that $\sigma(F, F)$ is well defined




- Do surgery on $W^{2q+1} \xrightarrow{F} X^{2q}$ to make $F$ a $q$-equivalence.
- $\Rightarrow H_i(W, M; \mathbb{Z}\pi)$ is (stably) free and is the only non-zero relative group. $i = q$ or $q+1$

$$\underset{\text{(Wall)}}{\Rightarrow} \quad W \cong (M \times I) \cup \Big(\underset{D^{q+1}\times D^q}{(q+1)\text{-handles}}\Big) \cup \Big(\underset{D^q \times D^{q+1}}{q\text{-handles}}\Big)$$

$\Rightarrow \exists$ an intermediate manifold $M_2 \subseteq W$ s.t.

$$M_2 \cong M_1 \#_r (S^q \times S^q)$$
$$M_2 \cong M_0 \#_s (S^q \times S^q)$$

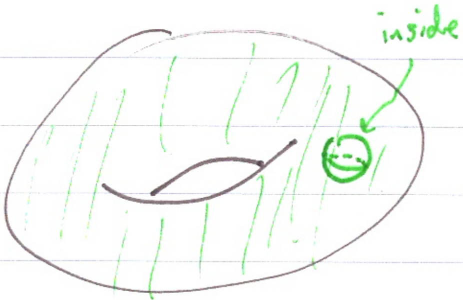
$$\Rightarrow (K_q(M_1), \lambda_1, \mu_1) \oplus H_\varepsilon(\mathbb{Z}\pi^r) \cong (K_q(M_2), \lambda_2, \mu_2) \oplus H_\varepsilon(\mathbb{Z}\pi^s)$$

$\therefore$ well def.

($\Rightarrow$) Choose $x_1, \dots, x_k \subset L \subseteq (K_q(M), \lambda, \mu)$ represented by
$$\coprod_{i=1}^{k} D^q \times S^q \hookrightarrow M$$

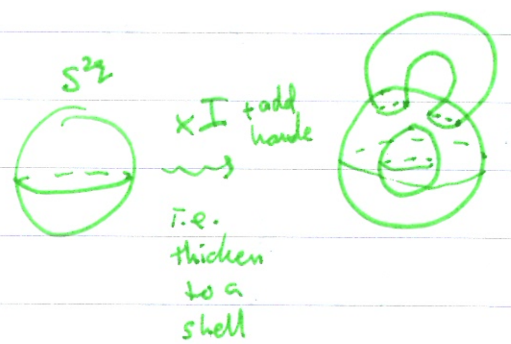
Perform surgery on embeddings $\Rightarrow$ htpy equiv.

<u>Picture</u> The trace of $S^q \times S^q \xrightarrow{\text{surger}} S^{2q}$




$$\text{Trace} = (S^q \times D^{q+1}) \setminus \operatorname{int}(D^{2q+1})$$

Look at it backwards.
Surger $S^{2q}$ to be $S^q \times S^q$
Attach handle to $S^{2q} \times I$ :



**Prop** $L_0(\mathbb{Z}) \cong 8\mathbb{Z}(\sigma) \cong \mathbb{Z}(E_8)$
$L_2(\mathbb{Z}) \cong \mathbb{Z}_2(A)$ $\qquad A = \left(\mathbb{Z}^2, \overset{\lambda}{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}, \overset{\mu}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}\right)$

$\square$

Lecture 6 — Diarmuid Crowley

## Odd dimensional surgery obstruction

$n = 2q+1$. Take a generating set $x_1, \ldots, x_k$ for $K_q(M)$ and represent by

$$\coprod_{i=1}^{k} D^{q+1} \times S^q \longrightarrow M$$

$\Rightarrow$ get a Heegard decomp $\quad M = U \cup_\partial M_0 \qquad \partial = \#_k S^q \times S^q$

$$\begin{array}{c} M = U \cup_\partial M_0 \\ \downarrow f \\ D^{2q+1} \cup X^\bullet \end{array}$$

$H_q(\partial U; \mathbb{Z}\pi) \cong H_\varepsilon(\mathbb{Z}\pi^k)$

We see 2 lagrangians

$K := \mathrm{Im}\left(H_{q+1}(U, \partial U) \longrightarrow H_\varepsilon(\mathbb{Z}\pi^k)\right)$
$L := \mathrm{Im}\left(H_{q+1}(M_0, \partial U) \longrightarrow H_\varepsilon(\mathbb{Z}\pi^k)\right)$

$\Lambda := \mathbb{Z}\pi$

$\Rightarrow \left(H_\varepsilon(\Lambda^k); K, L\right)$ a hyp form and 2 Lagrangians.

**Check** If $K+L = H_\varepsilon(\Lambda)$ then $f$ is a htpy equiv.

**Def'n** $L_{2q+1}(\mathbb{Z}\pi) := \left\{ (H_\varepsilon(\Lambda^k); K, L) \right\} / \sim$

where the formation is trivial if $H_\varepsilon = K+L$

- stable isometry $\Rightarrow$ $\sim 0$
- boundary formations $= 0$

Where $(H_\varepsilon; K, \cancel{L}\, \Gamma_\theta)$ is a boundary $\Gamma_\theta$ a graph for $\theta : L \to L^*$ a $(-\varepsilon)$-quad form.

where we have identified $H_\varepsilon = K \oplus K^*$.

**Theorem** $\sigma(\bar{f}, f) = 0 \Leftrightarrow (\bar{f}, f) \sim$ a htpy equiv.

**Prop** $L_{2q+1}(\mathbb{Z}) = 0 \quad \forall q$

$L_{2q+1}(\mathbb{Z}\pi) = 0$ for $\pi$ of odd order.

**Theorem** (Browder Novikov) $X$ a 1-conn PC with reducible SNF $n \geq 5$ ~~n odd~~

Then $X \simeq M$ a smooth manifold. $\quad n$ odd

$X \simeq M$ a PL manifold $\quad n$ even

~~and $X \simeq$~~

$X \simeq M$ for $M$ a smooth manifold $\quad n = 4k$

$\Leftrightarrow$ for some reduction $\xi$ of $\nu_X$

$$\sigma_X = \langle \mathcal{L}(-\xi), [X] \rangle$$

# The surgery exact sequence

**Th'm** ($s/h$ - cobordism) $n+1 \geq 6$.

(Smale - h) (Barden - Mazur Stallings - s)

$M_i^n \to W^{n+1}$ htpy equivs for $i = 1, 0$ (CAT mflds)

$\partial W = M_1 \sqcup \bar{M}_0$

$$\{W\}/_{\substack{\text{cat iso}\\ \text{rel } M_0}} \xrightarrow{\tau} Wh(\pi) \cong K_1(\mathbb{Z}\pi)/_{\{\pm g\}}$$

Then $\tau$ is a bijection with $\tau(0) = (M \times I, M \times 0, M \times 1)$

e.g. $Wh(e) = 0$, $Wh(\mathbb{Z}_5) = \mathbb{Z}$, $Wh(\pi) = 0$ if "$\pi$ negatively curved"

**Cor** (Poincaré Conjecture) $n \geq 6$ $\mathcal{S}^{CAT}(S^n) = \{id\}$ CAT = TOP or PL

**Proof** $\Sigma^n \simeq S^n \Rightarrow \Sigma^n \setminus (D^n \sqcup D^n) =: W$ an $h$-cob.
$\Rightarrow \Sigma^n \cong D^n \cup_f D^n$ some $f : S^{n-1} \to S^{n-1}$

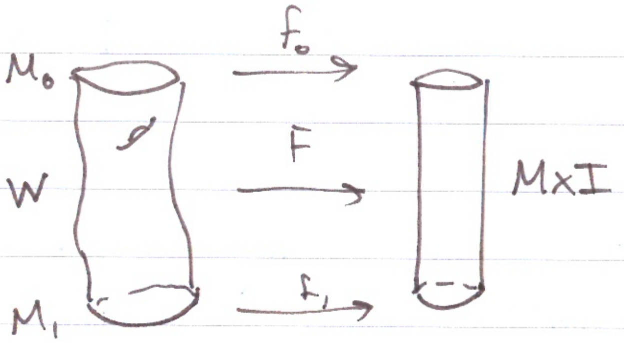
In CAT = TOP or PL we can cone $f$ to get $\Sigma^n \cong S^n$ (Alexander trick). □

Towards Wall realisation

**Def'n** Given $M$ a closed CAT manifold form

$$\omega(M, f_0) = \{(W, F)\}/\sim$$

Where




+ normal data

and $f_0, f_1$ are htpy equivalences.

<u>Theorem</u> (Wall realisation) $\exists$ well defined ~~bijective~~ surjective surgery obstruction map

$$\sigma : \mathcal{W}^{n+1}(M, f_0) \longrightarrow L_{n+1}(\mathbb{Z}\pi) \qquad \text{with } \ker(\sigma) = \text{h-cobordism}$$

<u>Proof</u> The surgery obstruction can be defined as before and we do all surgeries in the interior of $W$.

Suppose $\sigma(W, M_0, M_1; F) = 0$. As before $W$ is bordant rel $\partial$ to a htpy equivalence $W'$, an h-cobordism.

$\square$

<u>Simple htpy equivalences</u>

Two structure sets:
$$\mathcal{S}^h(M) := \{ N \overset{f}{\simeq} M \} / \text{h-cob}$$
$$\mathcal{S}^s(M) := \{ \underbrace{N \overset{f}{\simeq} M}_{\text{simple htpy equiv}} \} / \text{s-cob}$$

and $Wh(\pi) = 0 \Rightarrow$ agree.

**Theorem** (Browder-Novikov-Sullivan-Wall, Kirby-Siebenmann) There is a long exact sequence for $M^n$ a closed CAT manifold $n \geq 5$

$$\cdots \left[\Sigma M_+, G/CAT\right] \xrightarrow{\sigma} L_{n+1}(\mathbb{Z}\pi) \xrightarrow{\omega} \mathcal{S}^h(M) \xrightarrow{\eta} \left[M, G/CAT\right] \xrightarrow{\sigma} L_n(\mathbb{Z}\pi)$$

where $\eta(F: N \to M) := \left[\begin{tikzcd} \nu_N \arrow[r, "\bar{f}"] \arrow[d] & (f^{-1})^* \nu_N \arrow[d] \\ N \arrow[r, "f"] & M \end{tikzcd}\right]$

(Lück) $\Rightarrow$ canonical bundle extension $\bar{f}$ of $F$.

and

$$\omega(x \in L_{n+1}(\mathbb{Z}\pi)) = \partial_+(W; M_0, M_1; F) \text{ where } \sigma(W, -) = x$$

(Wall realisation)

<u>Wall realisation and plumbing</u>

Given $(P, \lambda, \mu)$ and $M^{2q-1}$ and gen set $x_1, \ldots, x_k$


Attach $q$-handles to $M$ along $x_1, \ldots, x_k$

<u>Example</u> $M = S^n$ $\quad$ CAT = PL, TOP $\quad$ $n \geq 6$

$$\begin{tikzcd}
\arrow[r] & L_{n+1}(\mathbb{Z}) \arrow[r] & \underbrace{\mathcal{S}^{CAT}(S^n)}_{=0} \arrow[r] & {[S^n, G/CAT]} \arrow[r] & L_n(\mathbb{Z}) \\
& \underbrace{\mathcal{S}^{CAT}(S^n \times I \text{ rel } \partial)}_{=0} \arrow[r] & {[\Sigma S^n_+, G/CAT]} \arrow[ul, bend left]
\end{tikzcd}$$

$$\Rightarrow L_{n+1}(\mathbb{Z}) \cong [\Sigma S^n_+, G/CAT] \cong \pi_{n+1}(G/CAT)$$

as $\Sigma S^n_+ \simeq S^{n+1} \vee S^1$ and $\pi_1(G/CAT) = 0$

<u>E.g.</u> $M = T^n$ or more generally any aspherical manifold.

<u>Conjecture</u> (Borel) $\mathcal{S}^{TOP}(M) = \{[\mathrm{id}_M]\}$

$$\begin{tikzcd}
\mathcal{S}^{TOP}(M) \arrow[r] & {[M, G/CAT]} \arrow[r] & L_n(\mathbb{Z}\pi) \\
& {[\Sigma M_+, G/CAT]} \arrow[r] & L_{n+1}(\mathbb{Z}\pi) \arrow[ull, bend left]
\end{tikzcd}$$

<u>n.b.</u> $[\Sigma M_+, G/CAT]$ is "computable" and $L_{n+1}(\mathbb{Z}\pi)$ are mysterious but fundamental.

For $M = T^n$ then $\Sigma T^n$ = big wedge of different dim spheres.

$$\Rightarrow [\Sigma T^n, G/CAT] \cong \bigoplus_j \pi_j(G/CAT)$$

$$\cong \bigoplus_j L_j(\mathbb{Z})$$

$$\Rightarrow L_i(\mathbb{Z}[\mathbb{Z}^n]) \cong \bigoplus L_{i_j}(\mathbb{Z})$$

ecture 7 — The TOP-GSES — Tiber

**Recall** $X$ a CAT $n$-manifold for CAT = PL, DIFF, TOP $n \geq 5$ $\pi = \pi_1 X$

(CAT-GSES)

$$\mathcal{N}_\partial^{CAT}(X \times I) \xrightarrow{\theta} L_{n+1}(\mathbb{Z}\pi) \xrightarrow{\partial} \mathcal{S}^{CAT}(X) \xrightarrow{\eta} \mathcal{N}^{CAT}(X) \xrightarrow{\theta} L_n(\mathbb{Z}\pi)$$

**Answer 2** (Uniqueness)

$$\begin{array}{l} f_0 : M_0 \xrightarrow{\simeq} X \\ f_1 : M_1 \xrightarrow{\simeq} X \end{array} \quad \begin{array}{c} \exists h : M_0 \xrightarrow{\cong} M_1 \\ \text{s.t. } f_1 \circ h \simeq f_0 \end{array} \iff [f_0] = [f_1] \in \mathcal{S}^{CAT}(X)$$

$$\iff \eta(f_0) = \eta(f_1) \in \mathcal{N}^{CAT}(X)$$
and $\exists\, (F, B) : (W, M_0, M_1) \longrightarrow (X \times I, X \times \{0\}, X \times \{1\})$
s.t. $\theta(F, B) = 0 \in L_{n+1}(\mathbb{Z}\pi)$

**Answer 1** (Existence)

$\theta : \mathcal{N}^{CAT}(X) \to L_n(\mathbb{Z}\pi)$

$X$ $n$-GPC $\exists M^n \simeq X \iff \mathcal{S}^{CAT}(X) \neq \emptyset \iff \mathcal{N}^{CAT}(X) = \emptyset$ and has non-triv kernel

Philosophy — "Uniqueness is the relative form of existence."

We want to interpret Answers 1, 2 in this framework i.e.

uniqueness $\iff$ "$0$" $= s(f_0, f_1) = [f_0] - [f_1] \in \mathcal{S}^{CAT}(X)$
existence $\iff$ "$0$" $= s(X) \in$ ???

$s$ is some sort of putative structure invariant.

This would make more sense if $\mathcal{S}^{CAT}(X) = \pi_{n+1}\left(\begin{smallmatrix}\text{some}\\ \text{space}\end{smallmatrix}\right)$. Then we could take $??? = \pi_n\left(\begin{smallmatrix}\text{some}\\ \text{space}\end{smallmatrix}\right)$.

<u>Question</u>: Can we specify the (CAT-GSES)?

<u>Answer (Quinn-Wall)</u>

Recall (CAT-GSES) can be extended to the left. Set $k \geq 0$.

$(\text{CAT-GSES})_k$

$$\mathcal{N}_\partial^{CAT}(X\times D^{k+1}) \xrightarrow{\theta} L_{n+k-1}(\mathbb{Z}\pi) \xrightarrow{\partial} \mathcal{S}_\partial^{CAT}(X\times D^k) \xrightarrow{\eta} \mathcal{N}_\partial^{CAT}(X\times D^k) \xrightarrow{\theta} L_{n+k}(\mathbb{Z}\pi)$$

where $\mathcal{S}_\partial^{CAT}(X\times D^k)$ is $\left[(f,\partial f) : (M,\partial M) \xrightarrow{(\simeq,\cong)} (X\times D^k, X\times S^{k-1})\right]$

where $[f_0,\partial f_0] \sim [f_1,\partial f_1]$ iff $\exists h : (M_0,\partial M_0) \xrightarrow{\cong} (M_1,\partial M_1)$ s.t. $f_1 \circ h \simeq f_0$ rel $\partial$.

<u>Th'm</u> $X$ - CAT $n$-manifold $n \geq 5$ then there exists fibration sequence of $\Delta$-sets

(CAT-GSFS) $\qquad \widetilde{\mathcal{S}}^{CAT}(X) \to \widetilde{\mathcal{N}}^{CAT}(X) \to \mathbb{L}_n(\mathbb{Z}\pi)$

fibration

s.t. $\pi_k(\text{CAT-GSFS}) = (\text{CAT-GSES})_k$

<u>Proof</u> (Idea) $\widetilde{\mathcal{S}}^{CAT}(X)^{(k)} := \left\{(f,\partial_i f) : \underbrace{(M,\partial_i M)}_{(n+k)\text{-dim CAT mfld "}(k+2)\text{-ad"}} \xrightarrow{(\simeq,\simeq)} (X\times\Delta^k, X\times\partial_i\Delta^k)\right\}$

- $\partial_i$ is given by restriction

3-ad 4-ad

Kan property $\Rightarrow \pi_k(\tilde{\mathcal{S}}^{CAT}(X)) \cong \mathcal{S}^{CAT}_\partial(X \times \Delta^k)$

$\tilde{\mathcal{N}}^{CAT}_\partial(X)^{(k)} := \{ (f,b) : (M, \partial_i M) \xrightarrow[\text{normal}]{\deg 1} (X \times \Delta^k, X \times \partial_i \Delta^k) \}$

Kan property $\Rightarrow \pi_k(\tilde{\mathcal{N}}^{CAT}_\partial(X)) \cong \mathcal{N}^{CAT}_\partial(X)$

$\mathcal{L}_n(\mathbb{Z}\pi)^{(k)} := \{ (f,b) : \underbrace{(M, \partial_i M)}_{(n+k)\text{-dim } (k+3)\text{-ad}} \xrightarrow[\text{normal}]{\deg 1} \underbrace{(Y, \partial_i Y)}_{(n+k)\text{-dim } (k+3)\text{-ad}} \}$

s.t. $(\partial_{k+2} f, \partial_{k+2} b) : \partial_{k+2} M \xrightarrow{\simeq} \partial_{k+2} Y$ a h.e.

and $r : Y \longrightarrow K(\pi, 1)$

Chapter 9 of Wall proves $\pi_k(\mathcal{L}_n(\mathbb{Z}\pi)) \cong L_{n+k}(\mathbb{Z}\pi)$

$\square$

Remark

(i) Th'm of Quinn-Wall is not quite satisfactory because $\mathcal{S}^{CAT}(X) = \pi_0(\tilde{\mathcal{S}}^{CAT}(X))$
$\Rightarrow$ no group structure in general.

(ii) We would be in a better position if we had spectra.

## L-Spectra

We have $\mathcal{L}_n(\mathbb{Z}\pi) \simeq \Omega \mathcal{L}_{n-1}(\mathbb{Z}\pi)$ not a typo

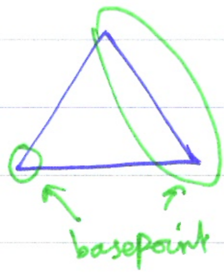
Def'n For $n \in \mathbb{Z}$ $\mathcal{L}_n(\mathbb{Z}\pi)^k := \begin{cases} \text{as before for } (n+k) \geq 0 \\ \{\emptyset\} \quad n+k < 0 \end{cases}$

__Corr__ The spaces $\mathcal{L}_{-n}(\mathbb{Z}\pi)$ form a spectrum $\mathbb{L}_\bullet(\mathbb{Z}\pi)$ s.t.

$$\pi_k(\mathbb{L}_\bullet(\mathbb{Z}\pi)) = L_k(\mathbb{Z}\pi).$$

__Proof__ of Th'm (Idea) Compare simplices

$$\mathcal{L}_n(\mathbb{Z}\pi)^{(k)} \subset \mathcal{L}_{n-1}(\mathbb{Z}\pi)^{(k+1)}$$




but our basepoint is taken to be $\emptyset$.

$\Downarrow$

. . .

$\square$

__What about $\mathcal{S}^{CAT}(X)$, $\mathcal{N}^{CAT}(X)$?__

$\mathcal{N}^{CAT}(X) \cong [X, G/CAT] \Rightarrow$ take H-space structure on $G/CAT$

__Prop__ (EXERCISE) $\theta : \mathcal{N}^{CAT}(X) \longrightarrow L_n(\mathbb{Z}\pi)$ is NOT a homo in general

Hint: even fails for simply connected

So this is a problem for general CAT.

# The case CAT = TOP

Th'm (Poincaré Conjecture) $\tilde{\mathcal{S}}^{TOP}(pt.) \simeq *$

Proof $\pi_n(\tilde{\mathcal{S}}^{TOP}(pt.)) = \mathcal{S}^{TOP}_\partial(D^n) = \mathcal{S}^{TOP}(S^n) = *$ by Poincaré Conjecture

$\square$

Cor $\eta : \mathbb{Z} \times G/TOP \xrightarrow{\simeq} \mathcal{L}_0\langle\mathbb{Z}\rangle$

Cor $\exists$ H-space structure on $G/TOP$ s.t. $\theta$ a homo

Proof Pull it back using $\eta$. $\square$

Cor $\exists$ fibration sequence of spectra

$$\underbrace{\text{(TOP-GSFS Spectra)}}\quad \tilde{\mathcal{S}}^{TOP}_\bullet(X) \longrightarrow \tilde{\mathcal{N}}^{TOP}_\bullet(X) \longrightarrow \mathcal{L}_{\bullet+1}\langle\mathbb{Z}\pi\rangle$$

s.t. $\pi_k(\quad) = (\text{TOP-GSES})_k$

$\square$

In particular

$$\mathcal{S}^{TOP}(X) \to \mathcal{N}^{TOP}(X) \to L_n(\mathbb{Z}\pi) \to \pi_{-1}(\mathcal{S}^{TOP}_\bullet(X)) \to \pi_{-1}(\mathcal{N}^{TOP}_\bullet(X))$$

and THIS is ??? from before – where the total surgery obstruction lives.

**Aim** For $X$ a l.f. simplicial cx define a fibration sequence of spectra

$$\mathbb{S}_\bullet(X) \longrightarrow \mathbb{L}_\bullet\langle 1\rangle(\mathbb{Z}_*(X)) \longrightarrow \mathbb{L}_\bullet(\mathbb{Z}\pi)$$

s.t. if $X$ an $n$-mfld $\exists$ maps

$$\begin{array}{ccccccc} \mathcal{S}^{TOP}(X) & \longrightarrow & \mathcal{N}^{TOP}(X) & \longrightarrow & L_n(\mathbb{Z}\pi) & & \\ \cong & & \cong & & \cong & & \\ \mathbb{S}_{n+1}(X) & \rightarrow & H_n(X;\mathbb{L}_\bullet\langle 1\rangle(\mathbb{Z})) & \rightarrow & \mathbb{L}_n(\mathbb{Z}\pi) & \longrightarrow & \mathbb{S}_n(X) \end{array}$$

Want to define $s(X) \in \mathbb{S}_n(X)$ and prove

$$s(X) = 0 \Leftrightarrow \exists\, M \text{ } n\text{-mfld s.t. } M \simeq X.$$

# The unreasonable effectiveness of cobordism in manifold theory

Andrew Ranicki

$\Omega_n$ = cobordism of $n$-manifolds

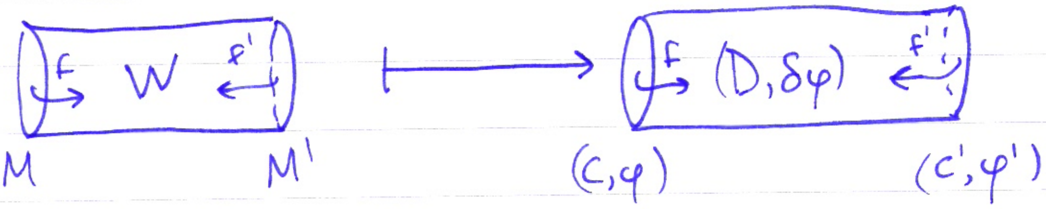
h/s-cobordism turn this cobordism into something "unreasonably effective".

What about an algebraic analogue?

$$\left\{ n\text{-dim Manifolds} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Chain cxes } C \text{ with} \\ n\text{-dim PD} \end{array} \right\}$$

$$M \longmapsto (C(M), \varphi)$$

where $\varphi_0 = [M] \cap - : C(M)^{n-*} \to C(M)$




Now note

$$\begin{array}{ccc} [M] \in H_n(M) & & [M'] \in H_n(M') \\ \searrow & & \swarrow \\ & f_*[M] = f'_*[M'] & \\ & \cap & \\ & H_n(W) & \end{array}$$

(maybe not on the chain level)

and $[W] \in H_{n+1}(W, \partial W) \longrightarrow ([M], [M']) \in H_n(\partial W) = H_n(M) \oplus H_n(M')$

So we mimic

$$\begin{tikzcd}
D^{n-*} \arrow[r, "f^*"] \arrow[d, equal] & C^{n-*} \arrow[r, "\varphi_0", "\cong"'] & C \arrow[r, "f_*"] & D \arrow[d, equal] \\
D^{n-*} \arrow[r, "f'^*"] & C'^{n-*} \arrow[r, "\varphi_0'", "\cong"'] & C' \arrow[r, "f'_*"] & D
\end{tikzcd}$$

Define $\delta\varphi_0 : \quad f_*\varphi_0 f^* \xrightarrow{\simeq} f'_*\varphi_0' f'^*$

Construct Poincaré-Lefschetz duality as a chain equivalence

$$\delta\varphi_0 \qquad D^{n+1-*} \xrightarrow{\simeq} \operatorname{cone}\big((f\,f') : C \oplus C' \longrightarrow D\big)$$

The function:

$$\begin{aligned} \Omega_n &\longrightarrow L^n(\mathbb{Z}) \\ M &\longmapsto (C(M), \varphi) \end{aligned}$$

Why $\mathbb{Z}$, not $\mathbb{Z}\pi$?

is a surjection.

<u>Theorem</u> Two symmetric Poincaré complexes $(C,\varphi)$ and $(C',\varphi')$ are chain htpy equivalent (i.e. $\exists\, h : C \xrightarrow{\simeq} C'$ s.t. $\exists\, \delta\varphi : h\varphi h^{-1} \xrightarrow{\simeq} \varphi'$) iff $\exists$ an algebraic $h$-cobordism (i.e. cobordism s.t. $f, f'$ chain equivs)

$$\big((f\ f') : C \oplus C' \longrightarrow D\big),\ (\delta\varphi,\ \varphi \oplus -\varphi')\big)$$

## Cartan + Eilenberg 1951 "Homological Algebra"

$G$-hypercohomology and $G$-hyperhomology with $G = \mathbb{Z}_2$

Let $V$ be a $\mathbb{Z}[\mathbb{Z}_2]$-module where $T : V \circlearrowleft$ $T^2 = id$.

$\mathbb{Z}_2$-hom
$$H^0(\mathbb{Z}_2; V) := \{ v \in V \mid Tv = v \} = \text{fixed points}$$
$$= \ker(1 - T : V \to V)$$

$\mathbb{Z}_2$-coho
$$H_0(\mathbb{Z}_2; V) := \operatorname{coker}(1 - T : V \to V) = \text{orbits of } T$$
$$= V / \{ v \sim Tv \}$$

Now $\mathbb{Z}_2$-hyper(co)homology:

Let $W_*$: $\dots \xrightarrow{1-T} \underset{W_2}{\mathbb{Z}[\mathbb{Z}_2]} \xrightarrow{1+T} \underset{W_1}{\mathbb{Z}[\mathbb{Z}_2]} \xrightarrow{1-T} \underset{W_0}{\mathbb{Z}[\mathbb{Z}_2]}$ be the standard free $\mathbb{Z}[\mathbb{Z}_2]$-module resolution of $\mathbb{Z}$. with $T = 1$.

$$H^n(\mathbb{Z}_2; V) = H_n(\operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, V))$$

where we see $v \in H^n(\mathbb{Z}_2; V)$ is $v : W \longrightarrow V_{*+n}$ a $\mathbb{Z}[\mathbb{Z}_2]$-module chain map

$$\begin{tikzcd}
W_2 \arrow[r, "v_2"] \arrow[d, "1+T"'] & V_{n+2} \arrow[d, "d"] \\
W_1 \arrow[r, "v_1"] \arrow[d, "1-T"'] & V_{n+1} \arrow[d, "d"] \\
W_0 \arrow[r, "v_0"] \arrow[d, "0"'] & V_n \arrow[d, "d"] \\
0 \arrow[r] & 0
\end{tikzcd}$$

s.t. $dv_0 = 0$
$dv_1 = (1-T)v_0$
$dv_2 = (1+T)$

$\Rightarrow \; v = [v_0, v_1, \ldots]$ (terminates for dimensional reasons)

$v \sim v'$ if $\exists\, u_0, u_1, \ldots \in V$ s.t.

$u_s \in V_{n+s+1}$ and $v_0 - v_0' = du_0$ (could regard $u_{-1} = 0$)

$v_1 - v_1' = du_1 + (1-T)u_0$

Recall: $\left[ \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,V)_n = \sum_{q-p=n} \mathrm{Hom}(W_p, V_q) \right.$

and if $f \in$ then $df := d_V f \pm f d_W$

(the differential of the Hom cx.) $\Big]$

---

$$H_n(\mathbb{Z}_2; V) := H_n(W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} V)$$

---

The $H_n$ and $H^n$ are related by a long exact sequence

$$\to H_n(\mathbb{Z}_2;V) \xrightarrow{1+T} H^n(\mathbb{Z}_2;V) \to \hat{H}^n(\mathbb{Z}_2;V) \to H_{n-1}(\mathbb{Z}_2;V) \to$$

Tate hypercoho.

---

Applying this to topology

For any space $X$ define the $\mathbb{Z}_2$-space $X \times X$ with $T =$ transpose.
and $V = C(X \times X)$

**Def'n** $Q^n(C(X)) := H^n(\mathbb{Z}_2; C(X \times X))$ $= H^n(\mathbb{Z}_2; C(X) \otimes C(X))$

$Q_n(C(X)) := H_n(\mathbb{Z}_2; C(X \times X))$ $= H_n(\mathbb{Z}_2; C(X) \otimes C(X))$

where we are careful about signs $T(x \otimes y) = \pm y \otimes x$

The Eilenberg-Zilber theorem gives a chain equivalence

$$\begin{tikzcd} C(X \times X) \arrow[r, "E_0", "\simeq"'] \arrow[d, "T"] & C(X) \otimes C(X) \arrow[d, "T"] \\ C(X \times X) \arrow[r, "\simeq", "E_0"'] & C(X) \otimes C(X) \end{tikzcd}$$

So this means $Q$ ~~the~~ should only depend on $X$.

DOES NOT COMMUTE

But $E_1 : E_0 T \xrightarrow{\simeq} T E_0$
$E_2 : E_1 T \xrightarrow{\simeq} T E_1$
$\vdots$
$E_{s+1} : E_s T \xrightarrow{\simeq} T E_s$

Note: This is how Steenrod squares defined after taking $C(X) \xrightarrow{\Delta} C(X \times X)$.

~~Furthermore~~ This can be viewed as the failure on the chain level of the cup product to commute.

$$\{E_s\} \subset H_0(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}_{\mathbb{Z}}(C(X \times X), C(X) \otimes C(X))))$$

**Def'n** The symmetric structure groups of a $\mathbb{Z}$-module chain cx are

$$Q^n(C) := H^n(\mathbb{Z}_2; C \otimes C)$$
$$= H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes C))$$

<u>Def'n</u> The quadratic structure groups of a $\mathbb{Z}$-module chain cx $C$ are

$$Q_n(C) := H_n(W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C \otimes C))$$

<u>Claim</u> For any space $X$ there is a natural transformation

$$\varphi : H_n(X) \longrightarrow Q^n(C(X))$$

<u>e.g.</u>

For any f.g. $\mathbb{Z}$-module $K$ define $C : \dots \to \underset{V_{i+1}}{0} \to \underset{V_i}{K^*} \to 0$

we can define $\varphi \in Q^{2i}(C)$ now just a symmetric form.

<u>Def'n</u> The quadratic structure groups of a $\mathbb{Z}$-module chain cx $C$ are

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ecture 9 — Zoo of structures on chain complexes — Andrew Ranicki

For any space $X$ with univ cover $\tilde{X}$ and $A = \mathbb{Z}\pi$ with $\omega$-twisted involution

1) Alexander-Steenrod-Steenrod — $\boxed{\text{SYMMETRIC CONSTRUCTION}}$

$$\varphi_X : H_n(X) \longrightarrow Q^n(C(\tilde{X}))$$
$$x \longmapsto (- \cap x) = \varphi_X(x)_0 + \text{others} \ldots$$

$$L^n(A) := \{(C, \varphi) \text{ s.t. } \varphi_0 : C^{n-*} \xrightarrow{\simeq} C_* \text{ chain equiv}\} / \text{cob}$$

$$\begin{array}{ccc} \Omega_n(K) & \ni & (M \xrightarrow{f} K) \\ \downarrow \sigma & & \downarrow \\ L^n(\mathbb{Z}\pi, K) & \ni & (C(\tilde{M}), \varphi_M[M]) \end{array}$$

SYMMETRIC SIGNATURE

Shouldn't this be a mapping cone?

But this is not an iso (it has a large kernel) e.g.

$$\Omega_n(K) = H_n(K; MSO) \longrightarrow L^n(\mathbb{Z}\pi, K))$$

Generalised Homology e.g. has MV

Not in general a homology e.g. no MV

2) $\boxed{\text{QUADRATIC CONSTRUCTION}}$

$$Q_n(C(X)) = H_n(S^\infty \times_{\mathbb{Z}_2} (X \times X))$$

Compare:

| | |
|---|---|
| $\varphi = \{\varphi_s \in (C \otimes_A C)_{n+s} \mid s \geq 0\}$ | $\psi = \{\psi \in (C \otimes_A C)_{n-s} \mid s \geq 0\}$ |
| $d\varphi_s = \varphi_{s-1} \pm T\varphi_{s-1}$ | $d\psi_s = \psi_{s+1} \pm T\psi_{s+1}$ |

and symmetrisation:

$$(1+T)\psi \in Q^n(C)$$

Th'm $\quad L_n(A) = \{(C,\psi) \text{ s.t. } (1+T)\psi_0 : C^{n-*} \to C_* \text{ chain equiv}\}/\text{cob}$

in the sense of Wall

Proof

(i) Identify $L_n(A)$ with the subgroup of $\{(C,\psi)\}$ that are

$$\cdots \to 0 \to C_k \to 0 \to \cdots \qquad n = 2k$$

$$\cdots \to 0 \to C_{k+1} \to C_k \to 0 \to \cdots \qquad n = 2k+1$$

(ii) Algebraic surgery below the middle dimension.

$\square$

How does a normal map $(\bar{f}, f) : M \to X$ determine an $n$-dim QAPC $(C,\psi)$ over $A$?

Want: $\overset{\text{WALL}}{L_n(\mathbb{Z}\pi, X)} \xrightarrow{\cong} \overset{\text{QAPC}}{L_n(A)}$

$$\sigma(f) \longmapsto (C, \psi)$$

What is this?

We have: $Th(\bar{F}) : Th(\nu_M) \longrightarrow Th(\xi) = Th(\nu_X)$

and S-duals:

$$Th(\bar{F})^* : \underset{\overset{\parallel}{X_+}}{Th(\nu_X)^*} \longrightarrow \underset{\overset{\parallel}{M_+}}{Th(\nu_M)^*}$$

inducing the chain-level Umkehr:

$$f^! : C(X) \xrightarrow{PD} C(X)^{n-*} \xrightarrow{F^*} C(M)^{n-*} \xrightarrow{PD} C(M)$$

We stabilise highly:

$$\boxed{F : \Sigma^\infty X_+ \longrightarrow \Sigma M_+}$$

STABLE GEOMETRIC UMKEHR

## Cartan Product Formula

$$\begin{array}{ccc}
H_m(X) \otimes H_n(Y) & \xrightarrow{\varphi_X \otimes \varphi_Y} & Q^n(C(X)) \otimes Q^n(C(Y)) \\
\downarrow \wr & & \downarrow \\
H_m(X \times Y) & \xrightarrow{\varphi_{X \times Y}} & Q^{m+n}(C(X) \otimes C(Y))
\end{array}$$

Should all be reduced?

in particular, take $Y = S^1$ and we know how to deal with suspensions.

To compare the $Q^*$ groups of $M$ and $X$ in $(F, f): M \to X$:

$$\begin{tikzcd} H_n(X) \cong H_{n+\infty}(\Sigma^\infty X_+) \arrow[r, "\varphi_{\Sigma^\infty X_+}"] \arrow[d, "F"] & Q^{n+\infty}(C(\Sigma^\infty X_+)) \arrow[d, "F \otimes F"] & Q^n(C(X)) \arrow[l, "S^\infty"'] \arrow[d, "F^! \otimes F^!"] \\ H_n(M) \cong H_{n+\infty}(\Sigma^\infty M_+) \arrow[r, "\varphi_{\Sigma^\infty M_+}"] & Q^{n+\infty}(C(\Sigma^\infty M_+)) & Q^n(C(M)) \arrow[l, "S^\infty"'] \end{tikzcd}$$

chain suspension

"cup products are 0 in suspension" ??

## The quadratic construction for a stable map

$F: \Sigma^\infty X_+ \to \Sigma^\infty M_+$ induces a natural transformation

$$\psi_F : H_n(X) \to Q_n(C(M)) \left( \xrightarrow{1+T} Q^n(C(M)) \right)$$ (To be defined later or not at all?)

This ~~is defined~~ is related to $\varphi_X, \varphi_M$ as follows:

$$F: \Sigma^\infty X_+ \to \Sigma^\infty M_+$$

$$\Rightarrow \quad f^! : C(X) \to C(M)$$

and $(1+T)\psi_F = \varphi_M f^! - (F^! \otimes f^!)\varphi_X$ i.e.

$$\begin{tikzcd} H_n(X) \arrow[r, "\varphi_X"] \arrow[d, "f^!"'] \arrow[dr] & Q^n(C(X)) \arrow[d, "f^! \otimes f^!"] \\ H_n(M) \arrow[r, "\varphi_M"] & Q^n(C(M)) \\ & Q_n(C(M)) \arrow[u] \end{tikzcd}$$

Now let $C(X) \xrightarrow{f^!} C(M) \xrightarrow{e} \mathrm{cone}(f^!)$

<u>Recall</u> ~~Theorem~~ Classically $H_*(\mathrm{cone}(f^!)) = K_*(M) = \ker(f_* : H_*(M) \to H_*(X))$

<u>Theorem</u> $(\mathrm{cone}(f^!), \psi = (e \otimes e)\psi_F[X])$ is an $n$-dim QAPC representing the Wall surgery obstruction

$$\begin{array}{ccc} L_n(\mathbb{Z}\pi, X) & \xrightarrow{\cong} & L_n(\mathbb{Z}\pi, X) \\ \sigma(F, f) & \longmapsto & (\mathrm{cone}(f^!), \psi) \end{array}$$

---

## <u>Lecture 10</u>

<u>Andrew Ranicki</u>

<u>Remark about $\psi_F$</u> (explicit construction)

The $\varphi$ map is actually induced by a chain map

CARTAN PRODUCT FORMULA

$$\begin{tikzcd} C(X \times Y) \arrow[rr, "\varphi_{X\times Y}"] \arrow[d] \arrow[drr, dashed] & & \mathrm{Hom}(W, C(X\times Y) \otimes C(X \times Y)) \arrow[d] \\ C(X) \otimes C(Y) \arrow[rr] \arrow[dr, "\varphi_X \otimes \varphi_Y"'] & & \mathrm{Hom}(W, (C(X) \otimes C(Y)) \otimes (C(X) \otimes C(Y))) \\ & \mathrm{Hom}(W, C(X) \otimes C(X)) \otimes \mathrm{Hom}(W, C(Y) \otimes C(Y)) \arrow[ur, "\Delta_W^*"'] & \end{tikzcd}$$

construct $\psi_F$ via this chain htpy.

This commutes up to chain htpy.

In particular take $Y = S^1 \Rightarrow \varphi_{\Sigma X_+} \simeq \varphi_{X_+} \otimes \varphi_{S^1}$

## 3 The Tate $\mathbb{Z}_2$-hyperecho

There is an exact sequence:

$$\cdots \to Q_n(C) \xrightarrow{1+T} Q^n(C) \xrightarrow{J} \hat{Q}^n(C) \to Q_{n-1}(C) \to \cdots$$

where $\hat{Q}$ defined by taking $\hat{W}: \cdots \to W_1 \xrightarrow{1-T} W_0 \xrightarrow{1+T} W^0 \xrightarrow{1-T} W^1 \to \cdots$

$$W_1 = \mathbb{Z}[\mathbb{Z}_2],\quad W_0 = \mathbb{Z}[\mathbb{Z}_2],\quad W^0 = \mathbb{Z}[\mathbb{Z}_2],\quad W^1 = \mathbb{Z}[\mathbb{Z}_2]$$

[Remark: $\hat{W}$ is the mapping cone of $1+T$ $\quad W^{-*} \xrightarrow{1+T} W \to \hat{W}$ ]

and defining

$$\hat{Q}^n(C) := \hat{H}^n(\mathbb{Z}_2; C\otimes C) = H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\hat{W}, C\otimes C))$$

(THESE GROUPS ARE 2-PERIODIC AS ~~THE~~ $\hat{W}$ is)

with elements $\hat{\varphi} = \{\hat{\varphi}_s \in (C\otimes C)_{n+s} \mid s \in \mathbb{Z}\}$

$$d\hat{\varphi}_s = \hat{\varphi}_{s-1} \pm T\hat{\varphi}_{s-1}$$

e.g. $\hat{Q}^n(\mathbb{Z}) = \mathbb{Z}_2$

**Theorem** $\hat{Q}^n(C) = \varinjlim_k Q^{n+k}(S^k C)$

$$= \lim\left(Q^n(C) \xrightarrow{S} Q^{n+1}(SC) \xrightarrow{S} Q^{n+2}(S^2C) \to \cdots\right)$$

**Corr** $\varphi \in \mathrm{im}\left(1+T : Q_n(C) \to Q^n(C)\right) \Leftrightarrow S^k\varphi = 0 \in Q^{n+k}(S^kC)$ for some $k \geq 0$.

<u>Claim</u> For $\Sigma^\infty X_+ \xrightarrow{F} \Sigma^\infty M_+$

$$\begin{tikzcd}
H_n(X) \arrow[d, "\psi_F"] \arrow[dr] \arrow[drr, "= 0 \,!!"] & & \\
Q_n(C(M)) \arrow[r, "1+T"] & Q^n(C(M)) \arrow[r, "J"] & \hat{Q}^n(C(M))
\end{tikzcd}$$

CLAIM

$$\psi = (\varphi, \chi) \qquad \varphi \qquad J\varphi = \partial\chi$$

quadratic refinement

Where \chi \in \Hom(\hat{W},C \otimes C). This because vanishes homology (i.e. in group) so it is boundary

QUESTION

What is the topological meaning of $H_n(X) \to Q^n(C(M)) \longrightarrow \hat{Q}^n(C(M))$

4 It depends on the $\boxed{\text{HYPERQUADRATIC CONSTRUCTION}}$ for an S-dual.

<u>Def'n</u> Let $S^N \longrightarrow Y \wedge Y^*$ be an S-dual so that $H_*(Y) \cong H^{N-*}(Y^*)$ then

$$\Theta_Y : H^k(Y) \longrightarrow \hat{Q}^{-k}(C^{-*}(Y))$$

Note: No $Y^*$ or $N$ here

constructed via $\left(H_{N-k}(Y^*) \xrightarrow{\varphi} Q^{N-k}(C(Y^*)) \xrightarrow{J} \hat{Q}^{N-k}(C^{N-*}(Y))\right)$

i.e. it is the symmetric construction for the S-dual where the choice made in taking an S-dual is cancelled out by the map $J$.

Now let $(\bar{f}, f) : M \to X$ a deg 1 normal map (n-dim)

$$\Rightarrow \quad Th(\bar{f}) : Th(\nu_M) \longrightarrow Th(\xi) = Th(\nu_X)$$

$$\Rightarrow \quad Th(\bar{f})^* : Th(\nu_X)^* \longrightarrow Th(\nu_M)^*$$

$$\Rightarrow \quad F : \Sigma^\infty X_+ \longrightarrow \Sigma^\infty M_+$$

Now use the hyperquadratic construction:

$$\begin{tikzcd}
F = f^! : H_n(X) = H^k(Th(\xi)) \arrow[r, "Th(\bar{F})^*"] \arrow[d, "\varphi_X"] \arrow[dd, "\theta_{Th(\xi)}"'] & H^k(Th(\nu_M)) = H_n(M) \arrow[d, "\varphi_M"] \arrow[dd, "\theta_{Th(\nu_M)}"'] \\
Q^n(C(X)) \arrow[d, "J"] & Q^n(C(M)) \arrow[d, "J"] \\
F \otimes F : \hat{Q}^n(C(X)) = \hat{Q}^{-k}(C(Th(\xi))^{-*}) \arrow[r, "\bar{F}^* \otimes \bar{F}^*"] & \hat{Q}^{-k}(C(Th(\nu_M))^{-*}) = \hat{Q}^n(C(M))
\end{tikzcd}$$

so that:

$$J\big((F \otimes F)\varphi_X - \varphi_M F\big) = \big(Th(\bar{F})^* \otimes Th(\bar{f})^*\big)\theta_{Th(\xi)} - \theta_{Th(\nu_M)} Th(\bar{F})^* = 0$$

THIS GIVES ANSWER TO QUESTION. This is claim

---

We now want to go straight to the surgery obstruction for a GPC $X$ (i.e. put the 2-stage obstruction into a single stage).

A $(k-1)$-spherical fibration $\nu_X : X \to BG(k)$ has a canonical hyperquadratic complex:

$$\hat{\sigma}^*(\nu) = \left( C(X),\ \theta(\nu) \in \hat{Q}^0(C(X)^{-*}) \right)$$

where $\theta(\nu) = \theta_{Th(\nu)}(U_\nu)$ for $U_\nu$ the Thom class

~~note that $U_\nu \in H^k(Th(\nu))$ and so it comes from $C^{-*}(X)$~~

then if $\nu_X : X \to BG(k)$ is the SNF then $J\sigma^*(X) = \hat{\sigma}^*(\nu_X)$

# Lecture 11 — Additive categories with chain duality and categories over complexes — Tibor Macko

**Recall** $R$ a ring with involution

SYMMETRIC SIGNATURE

$M$ an $n$-manifold $\quad \boxed{\mathrm{sign}^{\mathbb{L}^\bullet}_{\mathbb{Z}\pi}(M) := [(C(\tilde{M}), \varphi_M([M]))] \in L^n(\mathbb{Z}\pi)}$

$(f,b): M \to X$ deg 1 normal $\Rightarrow$ $C(\tilde{X}) \xrightarrow{f^!} C(\tilde{M}) \xrightarrow{e} \mathrm{cone}(f^!)$

$$\boxed{\mathrm{sign}^{\mathbb{L}_\bullet}_{\mathbb{Z}\pi}(f,b) := [(\mathrm{cone}(f^!), (e \otimes e)\psi_F([X]))] \in L_n(\mathbb{Z}\pi)}$$

QUADRATIC CONSTRUCTION

Need $L_n(\mathbb{A})$ for $\mathbb{A}$ an additive category

**Idea** $\mathbb{A}$ = additive cat

$\mathbb{B}(\mathbb{A})$ = bounded chain complexes over $\mathbb{A}$.

$C \in \mathbb{B}(\mathbb{A})$ but we want to make sense of $\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \underbrace{C \otimes_{\mathbb{A}} C}_{???})$

We want to mimic the case $R = \mathbb{A}$ and the slant map

$$-\backslash - : C \otimes_R D \xrightarrow{\cong} \mathrm{Hom}_R(C^{-*}, D)$$
$$x \otimes y \longmapsto (f \longmapsto \overline{f(x)} \cdot y)$$

---

"opposite category" reverses directions of morphisms

**Def'n** For $T: \mathbb{A}^{op} \to \mathbb{B}(\mathbb{A})$ define $T: \mathbb{B}(\mathbb{A})^{op} \to \mathbb{B}(\mathbb{A})$ by
$$T(C) = \mathrm{Tot}(T(C_p)_q)$$

**Def'n** Let $\mathbb{A}$ be an additive category. A <u>chain duality</u> on $\mathbb{A}$ is $(T, e)$

~~such that~~ where
- $T : \mathbb{A}^{op} \longrightarrow \mathbb{B}(\mathbb{A})$
- $e : T^2 \Rightarrow 1$ s.t. $\forall M \in \mathbb{A}$

(i) $e_M : T^2(M) \xrightarrow{\simeq} M$

(ii)
$$\begin{tikzcd} T(M) \arrow[r, "T(e_M)"] \arrow[rr, bend right, "1"'] & T^2(M) \arrow[r, "e_{T(M)}"] & T(M) \end{tikzcd}$$
commutes

<u>Def'n</u> $M \otimes_{\mathbb{A}} N := \mathrm{Hom}_{\mathbb{A}}(TM, N)$ $\Big\}$ chain complexes

$M, N \in \mathbb{A}$ $C \otimes_{\mathbb{A}} D := \mathrm{Hom}_{\mathbb{A}}(TC, D)$ $\Big\}$ of abelian groups.

$C, D \in \mathbb{B}(\mathbb{A})$

- $T_{M,N} : M \otimes_{\mathbb{A}} N \longrightarrow N \otimes_{\mathbb{A}} M$

$$\begin{array}{ccc} M \otimes_{\mathbb{A}} N & \longrightarrow & N \otimes_{\mathbb{A}} M \\ \| & & \| \\ \mathrm{Hom}_{\mathbb{A}}(TM, N) & & \mathrm{Hom}_{\mathbb{A}}(TN, M) \end{array}$$

$\boxed{\text{and this will be the involution for } M = N}$

by $(TM \xrightarrow{\varphi} N) \longmapsto (T(N) \xrightarrow{T(\varphi)} T^2(M) \xrightarrow{e_M} M)$

- $T_{C,D} : C \otimes_{\mathbb{A}} D \longrightarrow D \otimes_{\mathbb{A}} C$ with $(T_{C,D})_{p,q} = (-1)^{pq} T_{C_p, D_q}$

<u>Def'ns/Exercises/Prop</u>

- Can define $\hat{Q}^*_*(C)$ for $C \in \mathbb{B}(\mathbb{A})$
- SAPC, QAPC $\varphi_0 : \Sigma^n(TC) \xrightarrow{\simeq} C$
- $L_n(\mathbb{A}), L^n(\mathbb{A})$ are cobordism groups of $n$-dim SAPC and QAPC.

## <u>Functoriality</u>

<u>Def'n</u> A functor of additive categories with chain duality is a functor:

$F : \mathbb{A} \to \mathbb{A}'$ s.t. $\forall A \in \mathbb{A}$

- $G(A) : T' F(A) \xrightarrow{\simeq} FT(A)$.

- $$\begin{tikzcd} T'FT(A) \arrow[r, "G(TA)"] \arrow[d, "T'G(A)"'] \arrow[dr, phantom, "\circlearrowright"] & FT^2(A) \arrow[d, "F(e_A)"] \\ (T')^2 F(A) \arrow[r, "e'_{F(A)}"'] & F(A) \end{tikzcd}$$

and a (EXERCISE) this induces a map of L-theories

$$L^*_*(\mathbb{A}) \xrightarrow{L^*_*(F)} L^*_*(\mathbb{A}')$$

<u>Categories over complex</u>

Let $K$ be a l.f. simplicial complex, let $\mathbb{A}$ be an additive cat with chain duality.

$\Rightarrow \mathbb{A}^*(K), \mathbb{A}_*(K)$ ~~ad~~ cats with c.d.

2 <u>purposes</u>: $\mathbb{A}^*(\Delta^k)$ multiface duality
$\mathbb{A}_*(K)$ local duality for $K$ (or its failure

Suppose $K$ a triangulated manifold,
(i) $\forall \sigma \in K$, $\sigma$ is a $|\sigma|$-dim manifold w/ $\partial$
(ii) $\forall \sigma \in K$, $D(\sigma, K)$ is an $(n-|\sigma|)$-dim mfld w/ $\partial$.

<u>Def'n</u> $M \in \mathbb{A}$ is <u>$K$-based</u> if $M = \sum_{\sigma \in K} M(\sigma)$, $|\{\sigma \mid M(\sigma) \neq 0\}| < \infty$
A morphism of $K$-based objects
$f : M \to N$ is

$$f = \{ f(\tau,\sigma) : M(\sigma) \to N(\tau) \in \mathrm{Mor}(\mathbb{A}) \mid \sigma, \tau \in K \}$$

$\mathbb{A}^*(K)$ = obj K-based objs in $\mathbb{A}$
mor $\{ f = \{ f(\tau,\sigma) \mid f(\tau,\sigma) \neq 0 \Rightarrow \sigma \geq \tau \} \}$

$\mathbb{A}_*(K)$ = other way.

Explanation Chain cx in $\mathbb{B}(\mathbb{A}^*_*(K))$, $K = \Delta^1$ $\overset{\sigma_0}{\bullet} \overset{\tau}{\text{———}} \overset{\sigma_1}{\bullet}$, $C \in \mathbb{B}(\mathbb{A}^*(\Delta^1))$

$$\begin{tikzcd}
C_r : & C_r(\sigma_0) \arrow[d] & \oplus & C_r(\tau) \arrow[d] \arrow[dll] \arrow[drr] & \oplus & C_r(\sigma_1) \arrow[d] \\
C_{r-1} : & C_{r-1}(\sigma_0) & \oplus & C_{r-1}(\tau) & \oplus & C_{r-1}(\sigma_1)
\end{tikzcd}$$

: chain cxs.

Prop (Ranicki) (i) $C \in \mathbb{B}(\mathbb{A}^*_*(K))$ contractible $\Leftrightarrow$ $C(\sigma)$ is contr $\forall \sigma \in K$.
(ii) $f : C \to D$ in $\mathbb{B}(\mathbb{A}^*_*(K))$ chain equiv $\Leftrightarrow$ $f(\sigma,\sigma)$ is equiv $\forall \sigma \in K$.

Duality in $\mathbb{A}^*_*(K)$

We can construct K as a category with a morphism for each inclusion:
$\mathbb{A}^*[K]$ := ~~cova~~ contravariant functors $K \to \mathbb{A}$
$\mathbb{A}_*[K]$ := contravariant functors $K \to \mathbb{A}$

Def'n/Prop $(\mathbb{A}, T, e)$, $K$, define chain duality for $\mathbb{A}^*_*(K)$:

$$T_K : \mathbb{A}^*_*(K) \longrightarrow \mathbb{B}(\mathbb{A}^*_*(K))$$
$$\mathbb{A}^*_*(K) \longrightarrow \mathbb{A}^*_*[K] \longrightarrow \mathbb{B}(\mathbb{A}^*_*(K)) \longrightarrow \mathbb{B}(\mathbb{A}^*_*(K))$$
$$M \longmapsto ([M] : \sigma \to \sum_{\substack{\tau \leq \sigma \\ \geq}} M(\tau)) \longmapsto [M]^*[K] \longmapsto T_K(M)$$

$$T_K(M)_r(\sigma) = T\Big(\sum_{\substack{\tau \leq \sigma \\ \geq}} M(\tau)\Big)_{r \mp |\sigma|} \qquad S^{\pm|\sigma|}[M]$$

# Lecture 12 — Functoriality

Tibor Macko

**Recall** $\mathbb{B}(\mathbb{A}_*(K)) \longrightarrow \mathbb{B}(\mathbb{A}^*(K))$

$$C \longmapsto [C]_*[K](\sigma) = \sum_{\sigma \le \bar\sigma} S^{|\sigma|} C(\bar\sigma)$$

**Observation**:

$$[C]_*[K] = \sum_{\bar\sigma \in K} \left(\Delta(\Delta^{|\bar\sigma|}) \otimes C(\bar\sigma)\right)$$

**Prop'n** $\exists\, \beta_C : [C]_*[K] \xrightarrow{\simeq} C$ in $\mathbb{B}(\mathbb{A})$

$$a \otimes b \longmapsto \varepsilon(a) b$$

**Prop'n.** A simple map $f : J \to K$ induces a functor on the additive category w/ chain duality.

$$f^* : \mathbb{A}^*(K) \longrightarrow \mathbb{A}^*(J)$$
$$f_* : \mathbb{A}_*(J) \longrightarrow \mathbb{A}_*(K)$$ is called __assembly__

**Proof** Special case $K = pt. \Rightarrow f_* : \mathbb{A}_*(J) \longrightarrow \mathbb{A}$

$M \in \mathbb{A}_*(J)$, $f_*(M) \in \mathbb{A}$ and $\cancel{Tf_*(M)}$

$$G(M) : Tf_*(M) \xrightarrow{\simeq} f_* T_J(M) \quad ??$$

$$f_* T_J(M) = f_* T([M]_*[K]) = T(f_*([M]_*[K]))$$

$$\beta_M : [M]_*[K] \xrightarrow{\simeq} [M]$$
$$T\beta_M : TM \xrightarrow{\simeq} T([M]_*[K])$$

$\square$.

# Symmetric + Quadratic construction over K

**Recall** For $X$ $\quad \varphi : C(X) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C(\tilde{X}) \otimes_{\mathbb{Z}_\pi} C(\tilde{X}))$

and $\varphi_0([e]) = - \cap [c]$

So start with $K$ and $\Delta(K') \in \mathbb{B}(\cancel{\mathbb{A}}\, \mathbb{Z}_*(K))$. Let $C = C^{sing}(K')$ by $C(\sigma) = C^{sing}(D(\sigma, K), \partial D(\sigma, K))$.

Want $\varphi_K : C \longrightarrow \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \underbrace{C \otimes_{\mathbb{Z}_*(K)} C}_{:= \mathrm{Hom}_{\mathbb{Z}_*(K)}(TC, C)})$

want $\varphi_K(C)_0(\sigma) = - \cap \partial_\sigma c : \underset{\substack{\parallel \\ C^{n-|\sigma|}(D(\sigma))}}{\Sigma^k TC(\sigma)} \longrightarrow \underset{\substack{\parallel \\ C(D(\sigma), \partial D(\sigma))}}{C(\sigma)}$

---

$\forall \sigma\ \exists\ \partial_\sigma : C \to S^{|\sigma|} C(\sigma)$ a chain map

**Define**

$\sigma = \langle v_0, \ldots, v_{|\sigma|} \rangle$, $\quad \sigma_i := \langle v_0, \ldots, v_i \rangle$

$$\partial_\sigma : C_n = \Sigma\, C(\tau)_n \xrightarrow{\text{proj}} C(\sigma_0)_n \xrightarrow{d_1} C(\sigma_1)_{n-1} \xrightarrow{d_2} \cdots \to C(\sigma_{|\sigma|})_{n-|\sigma|}$$

$\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_{\mathbb{Z}_*(K)} C)$

$$M \otimes_{\mathbb{Z}_*(K)} N = \sum_{\sigma \in K} \sum_{\substack{\sigma \leq \\ \lambda, \mu}} S^{|\sigma|}(M(\lambda) \otimes_{\mathbb{Z}} M(\mu))$$

$$= \sum_{\sigma \in K} S^{|\sigma|} [M][\sigma] \otimes [N][\sigma]$$

$$C \simeq [C]_*[K] = \sum_{\sigma\in K} S^{|\sigma|} C(D(\sigma)) \dashrightarrow \sum_{\sigma\in K} S^{|\sigma|}\big(C(D(\sigma))\otimes C(D(\sigma))\big)$$

"fragmented version of Alexander-Whitney"

**Prop** Let $M^n$ $n$-mfld $r: M \to |K|$ s.t. $r \pitchfork D(\sigma, K)$. Then let $C = \sum C(\sigma)$ where $C(\sigma) = C(M(\sigma), \partial M(\sigma))$

$$M(\sigma) = r^{-1}(D(\sigma,K))$$

$$\mathrm{sign}_K^{\mathbb{L}^\bullet}(M) \in L^n(\mathbb{Z}_*(K))$$

ASIDE

For $K$ $\pi_1(K) \xrightarrow{\alpha} \pi$

**Def'n/Prop** $A_\alpha : \mathbb{Z}_*(K) \to \underbrace{\mathbb{Z}\pi\text{-modules}}_{\text{an additive category with chain duality}}$

$$M \longmapsto \sum_{\tilde\sigma\in\tilde K} M(p\tilde\sigma) \quad \text{for } p:\tilde K \to K.$$

The assembly map $A$ gives

$$L^n(\mathbb{Z}_*(K)) \xrightarrow{A} L^n(\mathbb{Z}\pi_1(K))$$

$$\mathrm{sign}_K^{\mathbb{L}^\bullet}(M) \longmapsto \mathrm{sign}_{\mathbb{Z}\pi}^{\mathbb{L}^\bullet}(M).$$

Let $(f,b)$ be deg 1 normal $M \to X$ and $r: X\to K$ with $r \pitchfork D(\sigma,K)$

Then

$$L_n(\mathbb{Z}_*(K)) \xrightarrow{A} L_n(\mathbb{Z}\pi_1(K))$$

$$\mathrm{sign}_K^{\mathbb{L}_\bullet}(f,b) \longmapsto \mathrm{sign}_{\mathbb{Z}\pi}^{\mathbb{L}_\bullet}(f,b)$$

# Generalised cohomology theories



## Bordism spaces of spectra

stable?

**Recall**: $\Omega_n^{SCAT}(K) := \{M \longrightarrow K\}/\text{bordism over } K.$

**Def'n** $(\Omega_n^{SCAT}(K))^{(k)} = \{(M, \partial_j M) \longrightarrow K \mid (M, \partial_j M) \text{ } (n+k)\text{-dim SCAT mfld } (k+2)\text{-ad}\}$

**Prop** (i) $\Omega_{n+k}^{SCAT}(K) \cong \pi_k(\Omega_n^{SCAT}(K))$

(ii) $\Omega_n^{SCAT}(K) \simeq \Omega\,\Omega_{n-1}^{SCAT}(K)$

---

**Def'n** For $n \in \mathbb{Z}$ define

$$\mathbb{L}^n(\mathbb{A})^{(k)} = \{n\text{-dim SAPCs in } \mathbb{B}(\mathbb{A}^*(\Delta^k))\}$$ "$(k+2)$-ads"

$$\mathbb{L}_n(\mathbb{A})^{(k)} = \{n\text{-dim QAPCs in } \mathbb{B}(\mathbb{A}^*(\Delta^k))\}$$

**Prop** $\pi_k(\mathbb{L}_n(\mathbb{A})) = L_{n+k}(\mathbb{A})$

$\pi_k(\mathbb{L}^n(\mathbb{A})) = L^{n+k}(\mathbb{A})$

and both $\mathbb{L}_n$ and $\mathbb{L}^n$ are spectra.

**Def'n/Prop** We have maps of spectra

"Poincaré bordism"

$$\mathrm{sign}^{\mathbb{L}^\bullet} : \Omega_\bullet^{STOP} \longrightarrow \Omega_\bullet^{P} \longrightarrow \mathbb{L}^\bullet$$

$$M \longmapsto \mathrm{sign}^{\mathbb{L}^\bullet}(M)$$

With no argument, understand $\mathbb{L}^\bullet(\mathbb{Z})$

symm sig over $\mathbb{Z}^*(\Delta^k)$

of spaces:

$$\operatorname{sign}^{\mathbb{L}^{\cdot}} : \underset{\widetilde{\mathcal{N}}^{TOP}(pt.)}{\overset{\parallel}{\operatorname{Sing}(G/TOP)}} \longrightarrow \mathbb{L}_0$$

$$\alpha \longmapsto (C, \psi)$$

which is defined by taking $\alpha : \Delta^{\ell} \to G/TOP$ to be represented by

$$(f(\alpha), \cancel{A}b(\alpha)) : (M, \partial_j M) \longrightarrow (\Delta^{\ell}, \partial_j \Delta^{\ell}) \in \widetilde{\mathcal{N}}^{TOP}(pt.).$$

and mapping to $(C_m, \psi)$ where $C(\sigma) = \operatorname{cone}(f(\alpha)(\sigma)^{!})$, quadratic construction over $\mathbb{Z}^*(\Delta^{\ell})$.

<u>Orientation Intermezzo</u>

X n-dim triangulated

$$\begin{array}{ccccc}
\mathcal{S}^{TOP}(X) & \longrightarrow & \mathcal{N}^{TOP}(X) & \longrightarrow & L_n(\mathbb{Z}\pi) \\
 & & \cong & & \\
 & & [X, G/TOP] & & \| \\
 & & \cong & & \\
 & & [X, \mathbb{L}_0\langle 1 \rangle] & & \downarrow \\
 & & \| & & \\
 & & \pi_0(\mathbb{L}_0\langle 1 \rangle^X) & \xrightarrow{\theta} & \pi_n(\mathbb{L}_\bullet(\mathbb{Z}\pi))
\end{array}$$

But properties:
- $\mathcal{N}^{TOP}(-)$ contravariant
- $L_n(-)$ covariant

On the other hand:

$$\operatorname{sign}_X^{\mathbb{L}^{\cdot}} : \mathcal{N}^{TOP}(X) \longrightarrow L_n(\mathbb{Z}_*(X))$$

<u>QUESTION</u> Isn't this a kind of PD?

## S-duality and P-duality

$\mathbb{E}$ an $\Omega$ spectrum of $\Delta$-sets $E_n \simeq \Omega E_{n-1}$, $K, J$ singular

$$H^n(K;\mathbb{E}) = [K_+, E_{-n}]$$
$$H_n(K;\mathbb{E}) = \pi_n(K_+ \wedge \mathbb{E})$$

S-duality: $\forall K\ \exists K^*$ with $\alpha: S^N \to K \wedge K^*$

$\boxed{\text{SLOGAN}}$ $H_*$ is $H^*$(S-dual)

$\exists$ a combinatorial description of S-duality $K \hookrightarrow \partial\Delta^{m+1}$

$$\begin{array}{ccc} (\partial\Delta^{m+1})^{m-\ell} & & (\Sigma^m)^{\ell} \\ \sigma & \longleftrightarrow & \sigma^* \\ \sigma \leq \tau & \longleftrightarrow & \sigma^* \geq \tau^* \end{array}$$

$K' \subseteq (\Sigma^m)'$ and define $\overline{K} \subset (\Sigma^m)'$ as

$\boxed{\text{Supplement}}$ $\overline{K} = \{\tau \in (\Sigma^m)' \mid \text{no face of } \tau \text{ is in } K'\}$

The idea is that $N/\partial N = \Sigma^m/\overline{K}$.

$N$ : regular n'hood

**Theorem** $\Sigma^m/\overline{K}$ is the S-dual of $K$. $\square$

**Cor** $H_n(K;\mathbb{E}) \cong H^{m-n}(\Sigma^m/\bar{K};\mathbb{E})$

$$\| $$

$$\Big\{ \big( \underset{\substack{\uparrow \\ (m-|\sigma|)\text{-dim}}}{\sigma^*} \longmapsto x(\sigma) \in (\mathbb{E}_{n-m})^{m-|\sigma|} \Big\}$$

$\square$

Now set $\mathbb{E} = \mathbb{L}_\bullet$ or $\mathbb{L}^\bullet$ $\Big($so $\mathbb{K}(\mathbb{L}_{n-m})^{m-|\sigma|}$ is $(n-|\sigma|)$-dim $(m-|\sigma|+2)$-ads$\Big)$

**Prop** (1) $\mathbb{L}_\bullet(\mathbb{A})^{K_+} \simeq \mathbb{L}_\bullet(\mathbb{A}^*(K))$

(2) $K_+ \wedge \mathbb{L}_\bullet(\mathbb{A}) \simeq \mathbb{L}_\bullet(\mathbb{A})^{\Sigma^m/\bar{K}} \simeq \mathbb{L}_\bullet(\mathbb{A}_*(K))$

**Proof** (1) LHS is $\sigma \mapsto 0$-dim $(|\sigma|+2)$-ad QAPC in $\mathbb{A}^*(\Delta^{|\sigma|})$
+ compatibility
$= 0$-dim QAPC in $\mathbb{A}^*(K)$.

(2) $\mathbb{A}_*(K) \cong \mathbb{A}^*(\Sigma^m/\bar{K})$

$\sigma \longleftrightarrow \sigma^*$

$\sigma \leq \tau \longleftrightarrow \sigma^* \geq \tau^*$

**Wrapping up the lecture**

**Def'n/Prop** $K$ a triangulated mfld.

(1) We have $\quad \mathrm{sign}_K^{\mathbb{L}^\bullet} : \Omega_n^{STOP}(K) \longrightarrow L^n(\mathbb{Z}_*(K)) = H_n(K;\mathbb{L}^\bullet)$.

(2) and $\quad \mathrm{sign}_K^{\mathbb{L}_\bullet} : \mathcal{N}^{TOP}(K) \longrightarrow L_n(\mathbb{Z}_*(K)) = H_n(K;\mathbb{L}_\bullet)$.

(3)
$$\begin{tikzcd}
\mathcal{N}^{TOP} \arrow[r, "\mathrm{sign}^{\mathbb{L}_\bullet}_{\mathbb{Z}\pi}"] \arrow[d, "\mathrm{sign}^{\mathbb{L}_\bullet}_K"'] & L_n(\mathbb{Z}\pi) \arrow[d, phantom, "\|"] \\
H_n(K;\mathbb{L}_\bullet) \arrow[r, "A"] & L_n(\mathbb{Z}\pi)
\end{tikzcd}$$
commutes

# L11: ADDITIVE CAT$^S$ WITH CHAIN DUALITY & CAT$^S$ OVER CPLXS

## 11.1.

Recall L8–L10: $R$–ring with involution $\rightsquigarrow$

$\rightsquigarrow$ $L_n(R)$ , $L^n(R)$ using chain cplxs

$\mathrm{sign}^{\mathbb{L}_\cdot}(f,b)$ , $\mathrm{sign}^{\mathbb{L}^\cdot}(M)$

solving problems (P1)+(P2) (L7)

AIM NOW: To describe the other terms from TOP-GSES as $L_*(\text{something})$ this will be an add. cat with chain duality (generalizing $R$-MOD)

IDEA: $\mathbb{A}$ = add. cat

$\mathbb{B}(\mathbb{A})$ = bounded chain cplxs in $\mathbb{A}$

$C \in \mathbb{B}(\mathbb{A})$ want $W^{\%}C = \mathrm{Hom}_{\mathbb{Z}_2}(W, C \otimes_{\mathbb{A}} C)$

need $C \otimes_{\mathbb{A}} D$ , $\mathbb{Z}_2 \curvearrowright C \otimes_{\mathbb{A}} C$

For $\mathbb{A} = R$-MOD have $-\backslash- : C \otimes_R D \xrightarrow{\cong} \mathrm{Hom}_R(\underset{\substack{\| \\ TC}}{C^{-*}}, D)$

$x \otimes y \longmapsto (f \mapsto \overline{f(x)} \cdot y)$

In $\mathbb{B}(\mathbb{A})$ we have $\mathrm{Hom}_{\mathbb{A}}(C, D)$,

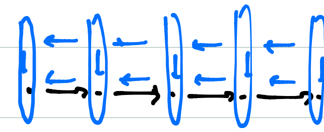
therefore we need $T : \mathbb{B}(\mathbb{A})^{op} \to \mathbb{B}(\mathbb{A})$ satisfying ...

then can hope to define $C \otimes_{\mathbb{A}} D = \mathrm{Hom}_{\mathbb{A}}(TC, D)$ ...

However $T$ will not be a strict inv., we need more flexibility

**DEF:** For $T: \mathbb{A}^{op} \to \mathbb{B}(\mathbb{A})$ define $T: \mathbb{B}(\mathbb{A})^{op} \to \mathbb{B}(\mathbb{A})$

by $T(C) = \mathrm{Tot}(T(\underset{\text{double cplx}}{C_{-p}})_q)$



**EXPL:** $\mathbb{A} = R\text{-MOD}$ $\quad T(M) = M^* \rightsquigarrow T(C) = C^{-*}$

**DEF:** Let $\mathbb{A}$ be an add. cat. A <u>chain duality</u> on $\mathbb{A}$ is a pair $(T, e)$ s.t.:

- $T: \mathbb{A}^{op} \to \mathbb{B}(\mathbb{A})$
- $e: T^2 \Rightarrow 1$ s.t.: • $e_M : T^2(M) \xrightarrow{\simeq} M$ — some flexibility, not too much
  - $1 = e_{TM} \circ T(e_M) : T(M) \to T^3(M) \to T(M)$

**EXPL:** $T(M) = M^*$ (but more interesting to come)

**DEF:**
$$M \otimes_{\mathbb{A}} N := \mathrm{Hom}_{\mathbb{A}}(T(M), N)$$
$$C \otimes_{\mathbb{A}} D := \mathrm{Hom}_{\mathbb{A}}(T(C), D)$$
chain cplxs of ab. grps covariant in both variables

**DEF/EXCS:** Def $T_{M,N} : \underset{\mathrm{Hom}_{\mathbb{A}}(TM,N)}{M \otimes_{\mathbb{A}} N} \to \underset{\mathrm{Hom}(TN,M)}{N \otimes_{\mathbb{A}} M}$

$$\varphi \longmapsto e_M \circ T(\varphi) : T(N) \to T^2(M) \to M$$

Show that $T_{M,N}$ is an iso.
Show that $T_{M,M}$ is an involution.

**COR/EXCS:** Def $T_{C,D} : C \otimes_{\mathbb{A}} D \to D \otimes_{\mathbb{A}} C$ by $(T_{C,D})_{p,q} = (-1)^{pq} T_{C_p, D_q}$
Show $T_{CC}$ is an involution.

DEF/PROP/EXCS:

For $C \in \mathbb{B}(\mathbb{A})$ define just as before

$W^{\%}C$, $W_{\%}C$, $\hat{W}^{\%}C$, $Q^n(C)$, $Q_n(C)$, $\hat{Q}^n(C)$

$S : \Sigma W^{\%}C \longrightarrow W^{\%}\Sigma C$, ..., $\hat{Q}^n(C) = \operatorname{colim}_k Q^{n+k}(\Sigma^k C)$

SAPC, QAPC $\varphi_0 : \Sigma^n TC \xrightarrow{\simeq} C$

pairs, cobordisms

$L^n(\mathbb{A})$, $L_n(\mathbb{A})$

get $1+T : L_n(\mathbb{A}) \longrightarrow L^n(\mathbb{A})$, 4-periodicity, ...

# FUNCTORIALITY:

**DEF:** A functor of add. cat. with chain duality is an add. functor $F: \mathbb{A} \to \mathbb{A}'$ s.t.

- $\forall A \in \mathbb{A}\ \exists\ G(A): T'F(A) \xrightarrow{\simeq} FT(A)$ nat. some flexibility
- $\forall A \in \mathbb{A}$

$$\begin{tikzcd}
T'FT(A) \arrow[r, "G(T(A))"] \arrow[d, "T'(G(A))"'] \arrow[dr, phantom, "\circlearrowleft"] & FT^2(A) \arrow[d, "F(e_A)"] \\
(T')^2F(A) \arrow[r, "e'_{F(A)}"] & F(A)
\end{tikzcd}$$

**PROP/EXCS:** $F: \mathbb{A} \to \mathbb{A}'$ induces $L^*_*(F): L^*_*(\mathbb{A}) \to L^*_*(\mathbb{A}')$

**PROOF:** $(C,\varphi)$ $\begin{array}{l} C \in \mathbb{B}(\mathbb{A}) \\ \varphi \in W^{\%}C_n \text{ cycle} \end{array}$ $\rightsquigarrow$ $\begin{array}{l} F(C) \in \mathbb{B}(\mathbb{A}') \\ F(\varphi) \in W^{\%}F(C)_n \text{ cycle?} \end{array}$

$\varphi \in Hom_{\mathbb{Z}_2}(W, C \otimes_{\mathbb{A}} C)$

Need a $\mathbb{Z}_2$-equiv. map $C \otimes_{\mathbb{A}} C \to F(C) \otimes_{\mathbb{A}'} F(C)$

Morphs $M \otimes_{\mathbb{A}} N \longrightarrow F(M) \otimes_{\mathbb{A}'} F(N)$

$$\left(T(M) \xrightarrow{\varphi} N\right) \longmapsto \left(T'F(M) \xrightarrow{G(M)} FT(M) \xrightarrow{F(\varphi)} F(N)\right)$$

Show the rest.

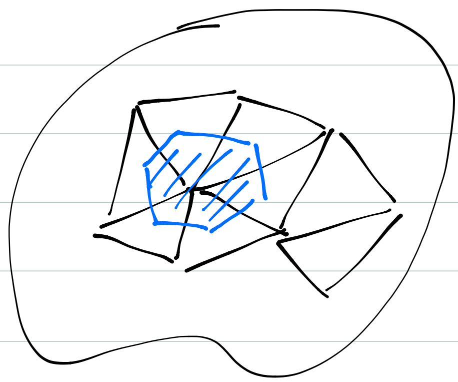
## 11.2: CAT OVER CPLXS

Here finally $\mathbb{A} \neq R$-MOD and the duality will be non-0-dim'

Let $K$ be a loc. fin. simpl. cplx
$\mathbb{A}$ add. cat. with chain duality $\Big\} \rightsquigarrow$ $\mathbb{A}^*(K)$, $\mathbb{A}_*(K)$

2 purposes: $\mathbb{A}^*(\Delta^k)$ multiface duality
$\mathbb{A}_*(K)$ local PD in $K$ (or its failure)

Motivation: $K$ = triangulated $n$-mfd



1) $\forall \sigma \in K$ $\sigma$ is $|\sigma|$-dim' mfd with $\partial_i \sigma$

2) $\forall \sigma \in K$ $D(\sigma, K)$ $(n-|\sigma|)$-dim' mfd with $\partial_\tau D(\sigma, K) = D(\tau, K)$ $\forall \sigma \leq \tau$

We would like to describe all these dualities by a single structured chain cplx

2 possibilities: $K$-based cplxs or functors (sheaves)

**DEF:** · $M \in \mathbb{A}$ is <u>$K$-based</u> if

$$M = \sum_{\sigma \in K} M(\sigma) \qquad |\{\sigma \in K \mid M(\sigma) \neq 0\}| < \infty$$

· A <u>morphism of $K$-based objects</u> is a collection

matrix

$$f : M \to N \qquad f = \{ f(\tau,\sigma) : M(\sigma) \to N(\tau) \in \mathrm{Mor}(\mathbb{A}) \mid \sigma, \tau \in K \}$$

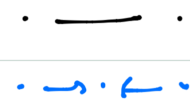
· <u>$\mathbb{A}^*(K)$</u>: obj: = $K$-based obj of $\mathbb{A}$

$$\mathrm{mor} = \{ f = \{ f(\tau,\sigma) \} \mid f(\tau,\sigma) \neq 0 \Rightarrow \sigma \geq \tau \} \}$$


$\Delta^1$

· <u>$\mathbb{A}_*(K)$</u>: obj: = — " —

$$\mathrm{mor} = \{ f = \{ f(\tau,\sigma) \} \mid f(\tau,\sigma) \neq 0 \Rightarrow \tau \geq \sigma \}$$


$\Delta^1$

**EXPL:** Chain cplxs in $\mathbb{B}(\mathbb{A}^*(K))$, $\mathbb{B}(\mathbb{A}_*(K))$

$K = \Delta^1$ $C \in \mathbb{B}(\mathbb{A}^*(\Delta^1))$

$$\begin{tikzcd}
 & \cdot\,\sigma_0 & & \overline{\tau} & & \cdot\,\sigma_1 \\
C_r \arrow[d, "d"] & C_r(\sigma_0) \arrow[d] & \oplus & C_r(\tau) \arrow[dll] \arrow[d] \arrow[drr] & \oplus & C_r(\sigma_1) \arrow[d] \\
C_{r-1} & C_{r-1}(\sigma_0) & \oplus & C_{r-1}(\tau) & \oplus & C_{r-1}(\sigma_1)
\end{tikzcd}$$

" $C = \{ \forall \sigma \in K \ (C(\sigma), d(\sigma,\sigma))$ is a chain cplx in $\mathbb{A} \}$ + more

EXPLS: • $\Delta(K)$ = simpl. chain cplx of $K \in \mathbb{B}(\mathbb{Z}^*(K))$

$$\Delta(K)(\sigma) = \Delta(\sigma, \partial\sigma) = S^{|\sigma|}\mathbb{Z}$$

$$\begin{tikzcd}
0 \arrow[d] & \mathbb{Z} \arrow[dl] \arrow[d] \arrow[dr] & 0 \arrow[d] \\
\mathbb{Z} & 0 & \mathbb{Z}
\end{tikzcd}$$

• $\Delta(K')$ = simpl. chain cplx of $K' \in \mathbb{B}(\mathbb{Z}_*(K))$

$\bullet$ —— $\bullet$ —— $\bullet$

$$\Delta(K')(\sigma) = \Delta(D(\sigma,K), \partial D(\sigma,K))$$

$$\begin{tikzcd}
\mathbb{Z} \arrow[d] \arrow[dr] & 0 \arrow[d] & \mathbb{Z}\oplus\mathbb{Z} \arrow[dl] \arrow[d] \arrow[dr] & 0 \arrow[d] & \mathbb{Z} \arrow[dl] \arrow[d] \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z}
\end{tikzcd}$$

PROP/DEF:

1) $C \in \mathbb{B}(\mathbb{A}_*^*(K))$ is contractible in $\mathbb{B}(\mathbb{A}_*^*(K)) \Leftrightarrow$
$\Leftrightarrow C(\sigma)$ is contractible in $\mathbb{A}$ $\forall \sigma \in K$

2) $f: C \to D$ in $\mathbb{B}(\mathbb{A}_*^*(K))$ is a chain equiv. in $\mathbb{B}(\mathbb{A}_*^*(K))$
$\Leftrightarrow \forall \sigma \in K$ $f(\sigma,\sigma): C(\sigma) \to D(\sigma)$ is a chain equiv. in $\mathbb{A}$

PROOF [RanWei'90]

# 11.3 DUALITY

Note: · Applying $T_{\mathbb{A}}$ would not work!

· Need to obtain dualities with $\partial$ of various dim's!

· The duality will be a composite of 3 fctors, need auxiliary cat's

$K$ = category, one morph. $\forall\, \sigma \le \tau$

$\mathbb{A}^*[K]$ = covariant fctors $K \longrightarrow \mathbb{A}$

$\mathbb{A}_*[K]$ = contravariant fctors $K^{op} \longrightarrow \mathbb{A}$

DEF/PROP: $(\mathbb{A}, (T, e))$, $K$ - simpl. cplx

$$T_K : \mathbb{A}^*_*(K) \longrightarrow \mathbb{B}(\mathbb{A}^*_*(K))$$

$$\mathbb{A}^*_*(K) \xrightarrow{\text{sum}} \mathbb{A}^*_*[K] \xrightarrow{\text{shift}} \mathbb{B}(\mathbb{A}^*_*(K)) \xrightarrow{T} \mathbb{B}(\mathbb{A}^*_*(K))$$

$$M \longmapsto [M] : \sigma \longmapsto \sum_{\substack{\tilde\sigma \le \sigma \\ \sigma \le \tilde\sigma}} M(\tilde\sigma) \longmapsto [M]^*_*[K] \longmapsto T_K M$$

$$[M]^*_*[K](\sigma) = S^{\overset{+}{-}|\sigma|}[M](\sigma)$$ check

$$T_K(M)_r(\sigma) = T\Big(\sum_{\substack{\tilde\sigma \le \sigma \\ \sigma \le \tilde\sigma}} M(\tilde\sigma)\Big)_{r \pm |\sigma|}$$

$d_{T_K M}$ - exes

<u>EXPL</u>: · $\Delta(K) \in \mathbb{B}(\mathbb{Z}^*(K))$ $\quad T_K(\Delta(K))(\sigma) = \Delta^{|\sigma|-1}(\sigma)$

· $\Delta(K') \in \mathbb{B}(\mathbb{Z}_*(K))$ $\quad T_K(\Delta(K'))(\sigma) = \Delta^{-*}(D(\sigma, K))$

$\bullet \rightarrow \bullet \leftarrow \bullet$ $\quad$ K as a cat.

$$\begin{tikzcd}
K=\Delta^1 & 1 & 0 \arrow[d] & \mathbb{Z} \arrow[dl] \arrow[d] \arrow[dr] & 0 \arrow[d] & \mathbb{Z}^*(K) \\
\Delta(K) \arrow[dd, "\text{shm}"] & 0 & \mathbb{Z} & 0 & \mathbb{Z} & \\
& 1 & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] & 0 \arrow[l] \arrow[d] & \mathbb{Z}^*[K] \\
{[\Delta(K)]} \arrow[dd, "\text{shift}"] & 0 & \mathbb{Z} \arrow[r] & \mathbb{Z}\oplus\mathbb{Z} & \mathbb{Z} \arrow[l] & \\
& 1 & 0 \arrow[d] \arrow[dr] & 0 \arrow[d] & 0 \arrow[d] \arrow[dl] & \mathbb{B}(\mathbb{Z}_*(K)) \\
{[\Delta(K)]^*[K]} \arrow[dd, "T_A"] & 0 & \mathbb{Z} \arrow[d] \arrow[dr] & \mathbb{Z} \arrow[d] & \mathbb{Z} \arrow[d] \arrow[dl] & \\
& -1 & 0 & \mathbb{Z}\oplus\mathbb{Z} & 0 & \\
& 1 & 0 \arrow[d] & \mathbb{Z}^*\oplus\mathbb{Z}^* \arrow[dl] \arrow[d] \arrow[dr] & 0 \arrow[d] & \mathbb{B}(\mathbb{Z}^*(K)) \\
T_K(\Delta(K)) & 0 & \mathbb{Z}^* & \mathbb{Z}^* & \mathbb{Z}^* &
\end{tikzcd}$$

PROOF:

Find $e_M : T^2 M \Rightarrow \mathbb{1}$

Hint: Under $\tau_{M,N} : M \otimes_A N \xrightarrow{\cong} N \otimes_A M$ for $N = TM$

we have

$$\tau_{M,TM} : \mathrm{Hom}_A(TM, TM) \to \mathrm{Hom}_A(T^2M, M)$$

$$\mathbb{1} \longmapsto e_M$$

For $A^*_*(K)$ find an obvious candidate for $\tau_{M,N}$ and define $e_M := \tau_{M,TM}(\mathbb{1})$ $\square$

## 11.4. FUNCTORIALITY

Recall: $\mathbb{A}_*(K) \xrightarrow{\text{sum + shift}} \mathbb{B}(\mathbb{A}^*(K))$

$$C \longmapsto [C]_*[K](\tau) = \sum_{\sigma \le \tau} S^{|\sigma|} C(\tau)$$

Observation: $[C]_*[K] = \sum_{\tau \in K} \left(\Delta(\Delta^{|\tau|}) \otimes C(\tau)\right)$

PROP: $\exists \beta_C : [C]_*[K] \xrightarrow{\simeq} C$ in $\mathbb{B}(\mathbb{Z})$

PROOF: $\sum_{\sigma \in K} \Delta(\Delta^{|\sigma|}) \otimes C(\sigma)_n \longrightarrow C = \sum_{\sigma \in K} C(\sigma)_n$

$$a \otimes b \longmapsto \varepsilon(a) \cdot b$$

augmentation

PROP: A simpl. map $f : J \to K$ induces a fctor of add. cat. ...

$$f^* : \mathbb{A}^*(K) \longrightarrow \mathbb{A}^*(J)$$
$$f_* : \mathbb{A}_*(J) \longrightarrow \mathbb{A}_*(K)$$

DEF: Special case $K = *$

$A = f_* : \mathbb{A}_*(J) \longrightarrow \mathbb{A}_*(*) = \mathbb{A}$ called assembly

is given by forgetting the K-based str.

PROOF: $f_*$, $K = *$ $\quad T_K = T_{A_*(K)}$, $T = T_{\mathbb{A}}$

$M \in \mathbb{A}_*(J)$ $\quad f_*(M) \in \mathbb{A}$ $\quad$ (forget $\Sigma$)

$G(M) : T f_*(M) \xrightarrow{\simeq} f_* T_J(M)$ ?

Notice $f_* T_J(M) = f_* T([M]_*[K])$ $\quad$ → here applying $f_*$ means there $\Sigma$

$\qquad\qquad = T f_*([M]_*[K])$ $\quad$ = sum + shift

We have $\beta_M : [M]_*[K] \xrightarrow{\simeq} M$

$G_M := T(\beta_M) : T(M) \xrightarrow{\simeq} T([M]_*[K])$

DEF/PROP: (Non 1-ctd assembly)

For $K$-simpl. cplx, $\pi_1(K) \xrightarrow{\alpha} \pi$, we obtain

$$A_\alpha : \mathbb{Z}_*(K) \longrightarrow \mathbb{Z}[\pi]$$
$$M \longmapsto \sum_{\tilde\sigma \in \tilde K} M(p\tilde\sigma)$$

where $p : \tilde K \to K$ is the covering induced by $\alpha$.

# 11.5 SYM. & QUAD CONSTR. OVER K

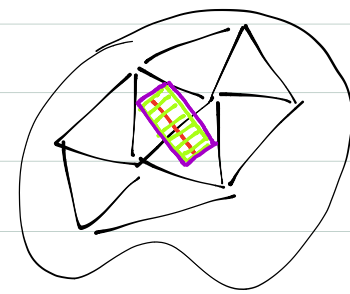
Recall: $X$ top. space $\rightsquigarrow$ $\varphi : C(\tilde{X}) \longrightarrow W^{\%} C(\tilde{X})$ chain map

Alexander-Whitney $\varphi(c)_0 = (- \cap c)$

$K$-simpl. cplx $\Delta(K') \in \mathbb{B}(\mathbb{Z}_*(K))$

$C = \Delta^{sing}(K') \in \mathbb{B}(\mathbb{Z}_*(K))$

$C(\sigma) = C^{sing}(D(\sigma), \partial D(\sigma))$



$\forall \sigma\ \exists\ \partial_\sigma : C \longrightarrow S^{|\sigma|} C(\sigma)$ as follows:

$$\partial_\sigma = C_n = \sum C(\tau)_n \xrightarrow{proj} C(\sigma_0)_n \xrightarrow{d_1} C(\sigma_1)_{n-1} \xrightarrow{d_2} \dots \xrightarrow{d_{|\sigma|}} C(\sigma)_{n-|\sigma|}$$

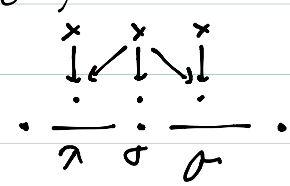
where $\sigma = \langle v_0 \dots v_{|\sigma|} \rangle$, denote $\sigma_i = \langle v_0 \cdots v_i \rangle$.

Want $\varphi_K : C \longrightarrow W^{\%} C$ chain map in $\mathbb{B}$

$(*)$ s.t. $\varphi_K(C)_0(\sigma) = (- \cap \partial_\sigma(c)) : \Sigma^n \underset{\substack{\parallel \\ C^{n-|\sigma|-*}(D(\sigma))}}{TC(\sigma)} \longrightarrow \underset{\substack{\parallel \\ C(D(\sigma), \partial D(\sigma))}}{C(\sigma)}$

Look at $W^{\%} C = \mathrm{Hom}_{\mathbb{Z}_2}(W, C \otimes_{\mathbb{Z}_*(K)} C)$

Claim: $(M \otimes_{\mathbb{Z}_*(K)} N)_r = \sum_{\sigma \in K} \sum_{\sigma \le \lambda, \mu} S^{|\sigma|}(M(\lambda) \otimes_{\mathbb{Z}} N(\mu))$



$(M \otimes_{\mathbb{Z}_*(K)} N) = \sum_{\sigma \in K} S^{|\sigma|}([M][\sigma] \otimes N[\sigma])$

$(C \otimes_{\mathbb{Z}_*(K)} C) = \sum_{\sigma \in K} S^{|\sigma|}([C][\sigma] \otimes [C][\sigma])$

**Note:** $[C][\sigma] = C(D(\sigma))$

**Recall:** $C \sim [C]_*[K] = \sum_{\sigma\in K} S^{|\sigma|}[C][\sigma] = \sum_{\sigma\in K} S^{|\sigma|}C(D(\sigma))$

$\forall \sigma$ we have $\varphi : C(D(\sigma)) \longrightarrow W^{\%}(C(D(\sigma)))$

$$S^{|\sigma|}C(D(\sigma)) \longrightarrow W^{\%}(S^{|\sigma|}C(D(\sigma)))$$

Together we obtain the desired

$$\varphi_K : C \simeq [C]_*[K] \longrightarrow W^{\%}C$$

**EXCS:** Check $(*)$.

**PROP:** Let $M^n$ be $n$-mfld, $r: M \to K$ map $r \pitchfork D(\sigma, K)$ $\forall \sigma \in K$

Then $\quad \operatorname{sign}_K^{\mathbb{L}^\bullet}(M) \in L^n(\mathbb{Z}_*(K))$

s.t. $A(\operatorname{sign}_K^{\mathbb{L}^\bullet}(M)) = \operatorname{sign}_{\mathbb{Z}}^{\mathbb{L}^\bullet}(M) \in L^n(\mathbb{Z})$

**PROOF:** Apply $\varphi_K([M])$

**PROP:** Let $(f,b): \overbrace{M \xrightarrow[\text{norm}]{\deg 1} X}^{n\text{-mflds}}$, $r: X \to K$, $r \pitchfork D(\sigma,K)$, $r\circ f \pitchfork D(\sigma,K)$

Then $\quad \operatorname{sign}_K^{\mathbb{L}_\bullet}(f,b) \in L_n(\mathbb{Z}_*(K))$

s.t. $A(\operatorname{sign}_K^{\mathbb{L}_\bullet}(f,b)) = \operatorname{sign}_{\mathbb{Z}}^{\mathbb{L}_\bullet}(f,b) \in L_n(\mathbb{Z})$

# L 12: GENERALIZED (CO-)HLGY THEORIES

## 12.1. Bordism spaces & spectra

Now want to exploit the "multiface duality" provided by $\mathbb{Z}^*(\Delta^k)$. Here the idea is classical and we talked about it in L7 already.

CAT = DIFF, PL, TOP, P (could even be N) $K$ space

Recall: $\Omega_n^{SCAT}(K) = \{ f : M \to K \mid M^n \text{ SCAT-mfd} \} / \sim$ ← cobordism of $M$ over $K$

DEF: $\mathbf{\Omega}_n^{SCAT}(K)^{(k)} = \{ f : (M, \partial_* M) \longrightarrow K \mid (M, \partial_* M)$ $(n+k)$-dim'l SCAT-mfd $(k+2)$-ad

$\emptyset$ SCAT $n$-mfd $\forall n$, base point.

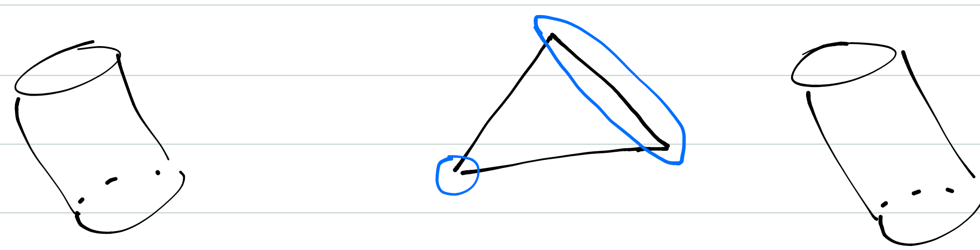
PROP: i) $\Omega_{n+k}^{SCAT}(K) \cong \pi_k(\mathbf{\Omega}_n^{SCAT}(K))$

ii) $\mathbf{\Omega}_n^{SCAT}(K) \simeq \Omega \mathbf{\Omega}_{n-1}^{SCAT}(K)$

PROOF: of (ii) $\mathbf{\Omega}_n^{SCAT}(K)^{(k)} \cong \Omega \mathbf{\Omega}_{n-1}^{SCAT}(K)^{(k)} \subset \mathbf{\Omega}_{n-1}^{SCAT}(K)^{(k+1)}$

$(n+k)$-dim'l $(k+2)$-ads

$((n-1)+(k+1))$-dim'l $(k+3)$-ads s.t. $\partial_{k+3} = \emptyset$, $\partial_0 \cdots \partial_{k+1} = \emptyset$



**DEF:** For $n \in \mathbb{Z}$ let  the dim$^n$ convention could be little confusing

$\mathbb{L}^n(\mathbb{A})^{(k)} = \{$ $n$-dim' SAPCs in $\mathbb{B}(\mathbb{A}^*(\Delta^k))\}$

$\mathbb{L}_n(\mathbb{A})^{(k)} = \{$ $n$-dim' QAPCs in $\mathbb{B}(\mathbb{A}^*(\Delta^k))\}$

**PROP:** $\mathbb{L}_n(\mathbb{A}) \simeq \Omega \mathbb{L}_{n-1}(\mathbb{A}) \quad \pi_k(\mathbb{L}_n(\mathbb{A})) = L_{n+k}(\mathbb{A})$

$\mathbb{L}^n(\mathbb{A}) \simeq \Omega \mathbb{L}^{n-1}(\mathbb{A}) \quad \pi_k(\mathbb{L}^n(\mathbb{A})) = L^{n+k}(\mathbb{A})$

**PROOF:** Exc: $\partial_i(C,\varphi) = (0,0)$ ...

**DEF/PROP:** We have maps of spectra

$\mathrm{sign}^{\mathbb{L}^\bullet} : \Omega^{STOP}_\bullet \longrightarrow \Omega^P_\bullet \longrightarrow \mathbb{L}^\bullet$ s.t.

$M \longmapsto \mathrm{sign}^{\mathbb{L}^\bullet}(M) = (C(M), \varphi[M])$

$\mathrm{sign}^{\mathbb{L}_\bullet} : G/TOP \xrightarrow{\simeq} \mathbb{L}_0\langle 1\rangle$ ← 1-conn. version

$\mathrm{Sing}(G/TOP) \simeq \mathcal{N}^{TOP}(*)$

$C(\sigma) = \mathscr{C}((f(\alpha)(\sigma))^!)$

$\alpha : \Delta^\ell \to G/TOP \longmapsto (C,\psi)$

$\wr$

$(f(\alpha), b(\alpha)) : (M(\alpha), \partial_i) \xrightarrow[\text{norm}]{\deg 1} (\Delta^\ell, \partial_i)$

$\ell$-dim$^l$ $(\ell+2)$-ads

**PROOF:** similar to sym/quad. constr. in $\mathbb{Z}_*(K)$ (now we have $\mathbb{Z}^\Delta(K)$)

## 12.2. Orientation intermezzo

Recall our goal: - to obtain alg. delooping of the TOP-GSES

Progress so far:

$$\begin{tikzcd} \mathcal{S}^{TOP}(X) \arrow[r] & \mathcal{N}^{TOP}(X) \arrow[r, "\theta"] \arrow[d, phantom, "\cong"] & L_n(\mathbb{Z}\pi) \arrow[d, leftrightarrow, "\cong"] \\ & {[X; G/TOP]} \arrow[d, phantom, "\cong"] & \pi_n(\mathbb{L}_\bullet(\mathbb{Z}\pi)) \\ & {[X; \mathbb{L}_0\langle 1\rangle]} \arrow[d, phantom, "="] & \\ & \pi_0(\mathrm{Maps}(X; \mathbb{L}_\bullet\langle 1\rangle)) \arrow[uur, "A"] & \end{tikzcd}$$

if $X$ n-mfd or at least $\exists$ reduction of SNF

mapping spectrum

In principle we could deloop the whole sequence now.

However, we would have technical problems defining TSO. Moreover, notice $\mathcal{N}^{TOP}$ - contravariant

$L_n$ - covariant

On the other hand, notice we have a map

$$\mathrm{sign}_X^{\mathbb{L}_\bullet} : \mathcal{N}^{TOP}(X) \longrightarrow L_n(\mathbb{Z}_*(X))$$

covariant, local

Q: Isn't this a kind of PD? We investigate this systematically

## 12.3. Hlgy & Cohlgy, S-duality & P-duality

$\mathbb{E}$ $\Omega$-spectrum of $\Delta$-sets $\mathbb{E}_n \simeq \Omega \mathbb{E}_{n-1}$
$K, L$ - simpl. cplxs loc. finite

$$H^n(K;\mathbb{E}) = [K_+ ; \mathbb{E}_{-n}] = \pi_{-n}(\mathbb{E}^{K_+}) \qquad (\mathbb{E}^{K_+})^{(p)} = \{K_+ \wedge \Delta^p \to \mathbb{E}\}$$

$$H_n(K;\mathbb{E}) = \pi_n(K_+ \wedge \mathbb{E}) = \operatorname{colim}_k \pi_{n+k}(K_+ \wedge \mathbb{E}_k)$$

S-duality: $\forall K \; \exists K^* \;\&\; \alpha : S^N \to K \wedge K^*$

$$\text{s.t. } \alpha \backslash - : H^n(K;\mathbb{E}) \xrightarrow{\cong} H_{N-n}(K^*;\mathbb{E})$$

If we work simplicially it turns out to be easier to work with cohlgy ($K \to \mathbb{E}$) than with hlgy ($S^N \to K_+ \wedge \mathbb{E}$) $\Rightarrow$

SLOGAN: $H_*$ is $H^*$(S-dual)

will have nice combinatorial description

# Combinatorial description of S-duality:

Let $K$ be a fin. simpl. cplx.

Choose $K \subset \partial\Delta^{m+1}$, $m \gg 0$

We first construct a simpl. cplx $\Sigma^m$ (nothing to do with $K$)

$$(\partial\Delta^{m+1})^{(m-\ell)} \xleftrightarrow{1-1} (\Sigma^m)^{(\ell)}$$

$$\sigma \longleftrightarrow \sigma^*$$

← this only works for $\partial\Delta^{m+1}$

$$\sigma \le \tau \longleftrightarrow \sigma^* \ge \tau^*$$

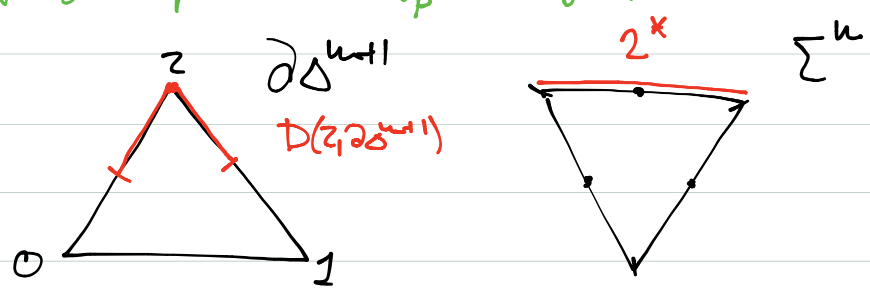
$$(\partial\Delta^{m+1})' \xrightarrow{\cong} (\Sigma^m)'$$

$$D(\sigma, \partial\Delta^{m+1}) \longleftrightarrow (\sigma^*)'$$

$\hat\sigma_0 \dots \hat\sigma_p$ $\qquad$ $\hat\sigma_0^* \dots \hat\sigma_p^*$

$\sigma = \sigma_0 < \dots < \sigma_p$ $\qquad$ $\hat\sigma_p^* < \dots < \hat\sigma_0^* = \sigma^*$

$m=1$



**DEF:** Let $\bar{K} \subset \Sigma^m$ be the subcomplex of $\Sigma^m$ given by

$$\bar{K} = \{\sigma' \in (\Sigma^m)' \mid \text{no face of } \sigma' \text{ is in } K'\}$$

**THM** (Whitehead)
$\Sigma^m/\bar{K}$ is an S-dual to $K$

**COR:** $H_n(K;\mathbb{E}) \cong H^{m-n}(\Sigma^m/\bar{K};\mathbb{E}) = H^{m-n}(\Sigma^m,\bar{K};\mathbb{E})$

$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ x & \longmapsto & x : \sigma^* \mapsto x(\sigma) \in (\mathbb{E}_{n-m})^{(m-|\sigma|)} \\ & & \text{in } (\Sigma^m)^{(m-|\sigma|)} \end{array}$$

**REM:** $\mathbb{E} = \mathbb{L}^\bullet$ or $\mathbb{L}_\bullet$. $\quad (\mathbb{E}_{n-m})^{(m-|\sigma|)}$ $(n-|\sigma|)$-dim'l
$(m-|\sigma|+2)$-ads

Think of $Z_*(X)$

## 12.4. Co-hlgy with $\mathbb{L}$-spectra

PROP:

1) $\mathbb{L}_\bullet(\mathbb{A})^{K_+} \simeq \mathbb{L}_\bullet(\mathbb{A}^*(K))$

2) $K_+ \wedge \mathbb{L}_\bullet(\mathbb{A}) \simeq \mathbb{L}_\bullet(\mathbb{A})^{\Sigma^m/K} \simeq \mathbb{L}_\bullet(\mathbb{A}_*(K))$

PROOF: ([Ran92, 13.7], [LM09, 14.2])

be careful about dim's before

1) 0-simplex of LHS $= \sigma \mapsto (0)$-dim$^l$ $\mathbb{A}$-ad

0-dim$^l$ QAPC in $\mathbb{A}^*(\Delta^{|\sigma|})$

+ compatibility

$=$ 0-dim$^l$ QAPC in $\mathbb{A}^*(K)$

2) $\mathbb{A}_*(K) \cong \mathbb{A}^*(\Sigma^m/K)$

$\sigma \longleftrightarrow \sigma^*$

$\sigma \leq \tau \longleftrightarrow \sigma^* \geq \tau^*$

## 12.5. Wrap-up

DEF/PROP: $K$ – fin. simpl. cplx

1) We have $\mathrm{sign}_K^{\mathbb{L}^\bullet}: \Omega_n^{STOP}(K) \longrightarrow H_n(K;\mathbb{L}^\bullet) = L^n(\mathbb{Z}_*(K))$

$\searrow H_n(K;\Omega_\bullet^{STOP}) \nearrow$

$r: M^n \to K$ gives $\sigma \mapsto M(\sigma)$

$r \pitchfork D(\sigma,K)$ $\quad (n-|\sigma|)$-dim'l

$M(\sigma) = r^{-1}(D(\sigma,K))$ $\quad$ mfld with $\partial$

$M = \bigcup_{\sigma\in K} M(\sigma)$ $\quad \rightsquigarrow$

$\sigma \mapsto (C,\varphi)$ in $\mathbb{B}(\mathbb{Z}_*(K))$

with $C(\sigma) = C(M(\sigma),\partial M(\sigma))$

$-\cap[M(\sigma)] = \varphi(\sigma,\sigma): C^{n-|\sigma|-*}(M(\sigma)) \to C(M(\sigma),\partial M(\sigma))$

2) For $r: \overset{n\text{-mfld}}{X^n} \to K$ we have

$\mathrm{sign}_K^{\mathbb{L}_\bullet}: \mathcal{N}^{TOP}(X) \longrightarrow H_n(K;\mathbb{L}_\bullet) = L_n(\mathbb{Z}_*(K))$

$r \pitchfork D(\sigma,K)$ $\quad \sigma \mapsto$

$(f,b): M \xrightarrow[\text{norm}]{\deg 1} X$ $\quad \rightsquigarrow \quad (f(\sigma),b(\sigma)): (M(\sigma),\partial) \xrightarrow[\text{norm}]{\deg 1} (X(\sigma),\partial)$

$f \pitchfork X(\sigma) = r^{-1}(D(\sigma,K))$ $\quad$ of $(n-|\sigma|)$-dim'l mflds with $\partial$

$M(\sigma) = f^{-1}(X(\sigma))$ $\quad \rightsquigarrow$ quad refinement of above

$\sigma \mapsto (C,\psi)$ in $\mathbb{B}(\mathbb{Z}_*(K))$

with $C(\sigma) = \mathcal{C}(f(\sigma)^!)$

$\psi(\sigma,\sigma) = \psi(f(\sigma)^!)$

3) $A(\mathrm{sign}^{\mathbb{L}^\cdot}_K(M)) = \mathrm{sign}^{\mathbb{L}^\cdot}_{\mathbb{Z}\pi}(M) \in L^n(\mathbb{Z}\pi)$

4) $A(\mathrm{sign}^{\mathbb{L}_\cdot}_K(f,b)) = \mathrm{sign}^{\mathbb{L}_\cdot}_{\mathbb{Z}\pi}(f,b) \in L_n(\mathbb{Z}\pi)$

$$\begin{tikzcd}
\mathcal{N}^{TOP}(X) \arrow[r, "\mathrm{sign}^{\mathbb{L}_\cdot}_{\mathbb{Z}\pi} = \theta"] \arrow[d, "\mathrm{sign}^{\mathbb{L}_\cdot}_X"', "K \text{ does do } \cong"] \arrow[dr, phantom, "\circlearrowright"] & L_n(\mathbb{Z}\pi) \arrow[d, "="] \\
H_n(X;\mathbb{L}_\bullet) \arrow[r, "A"] & L_n(\mathbb{Z}\pi)
\end{tikzcd}$$

5) The bottom row is defined for any simplicial cplx $X$ (not nec. $n$-mfd) and is covariant (both terms

Lecture 12 — The Normal Complexes — Tibor Macko

## Recall

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{S}^{TOP}(X) \arrow[r] \arrow[d, "\text{close to } \cong"] & \mathcal{N}^{TOP}(X) \arrow[r] \arrow[d, "\text{close to } \cong"] & L_n(\mathbb{Z}\pi) \arrow[r] \arrow[d, "="] & \cdots \\
\cdots \arrow[r] & \mathbb{S}_{n+1}(X) \arrow[r] & H_n(X, \mathbb{L}_\bullet) \arrow[r, "A"] & L_n(\mathbb{Z}\pi) \arrow[r] & \mathbb{S}_n(X) \arrow[r] & \cdots
\end{tikzcd}$$

can be just the htpy fibre

mixture — local — global — mixture

"We have de-looped the surgery exact sequence."

We expect the total surgery obstruction to live here.

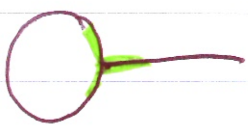
**Question** What local structure does an $n$-dimensional GPC have?

**Answer** It is locally a geometric normal complex.

## Normal Geometric Poincaré Complexes ($n$-GPC's)

**Def'n** $n$-GPC is a triple $(Y, \nu, \rho)$
- $Y$ a l.f. simplicial cx
- $\nu : Y \to BSG(k)$ $(k-1)$-dim spherical fib
- $\rho : S^{n+k} \to Th(Y)$

**Remark** Not locally Poincaré



**E.g.** $X$ a GPC $\Rightarrow$ $(X, \nu_X, c)$ a $n$-GPC.

**Def'n** An $(n+1)$-GNP is $((Z,Y),\nu,\rho)$ s.t.

- $(Z,Y)$ l.f. simplicial pair
- $\nu : Z \to BSG(k)$, $\nu|_Y : Y \to BSG(\ )$
- $\rho : (D^{n+k+1}, S^{n+k}) \to (Th(\nu), Th(\nu|_Y))$

An $(n+1)$-geom normal cobordism $(X,\nu,\rho)$, $(X',\nu',\rho')$ is $(n+1)$-GNP $((Z, X \amalg X'), \nu'', \rho'')$ s.t.

$$\nu''|_X = \nu,\quad \nu''|_{X'} = \nu',\quad \rho''|_{S^{n+k}_+} = \rho,\quad \rho''|_{S^{n+k}_-} = \rho'$$


**Def'n** $\Omega^N_n(K) = \{(X,\nu,\rho)\ n\text{-GPC},\ f : X \to K\}/\text{cob}$

$$\Omega^N_n(K)^{(k)} := \left\{ (X,\nu,\rho) \;\middle|\; \begin{array}{l} X \text{ simp } (k+2)\text{-ad} \\ \nu : X \to BSG(r) \quad \rho : \Delta^{n+k+r} \to Th(\nu) \\ \quad \text{s.t. } \rho(\partial_i \Delta^{n+k+r}) \subseteq Th(\nu|_{\partial_i X}) \\ \quad\quad i = 0,1,\dots,k. \end{array} \right\}$$

$\Omega^N_\bullet = \{\Omega^N_{-n}\}_n$ is an $\Omega$-spec.

**Def'n/Prop** $X$ $n$-GPC. Then there exists

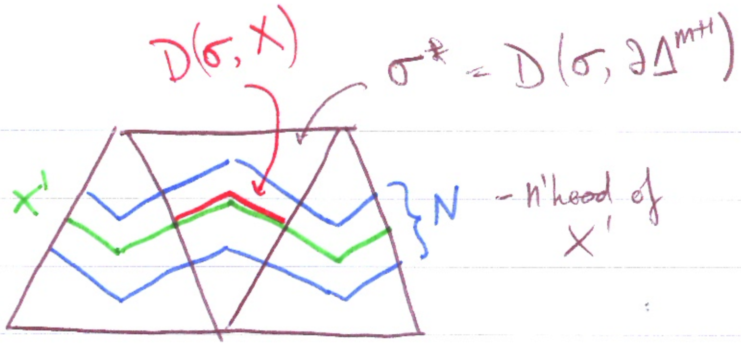
$$\text{sign}^{\Omega^N_\bullet}_X(X) \in H_n(X; \Omega^N_\bullet)$$
$$\ni$$
$$(\sigma \mapsto (X(\sigma), \nu(\sigma), \rho(\sigma)) \in (\Omega^N_{n-m})^{(m-|\sigma|)}$$

$(n-|\sigma|)$-dim
$(m-|\sigma|+2)$-ad

**Proof** Choose $X \subset \partial\Delta^{m+1}$

Then $$\Sigma^m \xrightarrow{\rho_X} Th(\nu_X) \simeq \Sigma^m/\bar{K}$$
$$\sigma^* \in \Sigma^m$$

$\dim(\sigma^*) = m - |\sigma|$




$X(\sigma) = D(\sigma, X)$

$\nu(\sigma) = \nu_X|_{X(\sigma)}$

$\rho(\sigma) = \text{Pont-Thom}$

$\sigma^* \mapsto Th(\nu(\sigma))$

$$\begin{tikzcd}
N(\sigma) = \sigma^* \cap N \arrow[rr, dashed] \arrow[dr, dashed] \arrow[ddr] & & N \arrow[d] \\
& D(\nu(\sigma)) \arrow[r] \arrow[d] & D(\nu_X) \arrow[d] \\
& X(\sigma) \arrow[r] & X
\end{tikzcd}$$

commutes.

$\square$

## Extracting algebra from a normal ex.

**Recall** $X$ a $n$-GPC, $\nu_X$ and $\rho_X : S^{n+k} \to Th(\nu_X)$.

$$\alpha_X : S^{n+k} \longrightarrow Th(\nu_X) \longrightarrow X_+ \wedge Th(\nu_X)$$

$$\begin{tikzcd}
C^{n-*}(X) \arrow[rr, "- \cup U(\nu)"] \arrow[dr, "- \cap {[X]}"'] & & C^{n+k-*}(Th(\nu_X)) \arrow[dl, "\alpha_X \backslash -"] \\
& C_*(X) &
\end{tikzcd}$$

commutes.

Now $X$ a $n$-GPC so we can define $[X] := h(\rho) \cap U(\nu)$.

BUT $- \cap [X] : C^{n-*} \to C$ $\boxed{\text{not} \simeq \text{any more!}}$

"$n$-GPC is a wannabe $n$-GPC".

We still have S-dual:

maybe no longer $X_+$

$$\alpha_{Th(\nu)} : S^N \longrightarrow Th(\nu) \wedge Th(\nu)^*$$

but $\rightsquigarrow$ ∄

$$\Delta \circ \rho : S^{n+k} \longrightarrow Th(\nu) \longrightarrow X_+ \wedge Th(\nu)$$

can suspend so that $N = n+k+p$:

$$\Sigma^p(\Delta\circ\rho) : S^{\overbrace{n+k+p}^{=N}} \longrightarrow \Sigma^p Th(\nu) \longrightarrow \Sigma^p X_+ \wedge Th(\nu)$$

$$\Rightarrow \quad \alpha_{Th(\nu)} \backslash - : [Th(\nu)^*, \Sigma^p X_+] \xrightarrow{\cong} [S^N, \Sigma^p X_+ \wedge Th(\nu)]$$

Define $\longrightarrow$ $\Gamma_x \longleftrightarrow \Delta\circ\rho$.

look at

$$\begin{tikzcd}
C^{n-*}(X) \arrow[r, "-\cup U", "\simeq"'] \arrow[d, "-\cap[X]"'] & C^{n+k-*}(Th(\nu_X)) \arrow[r, "\alpha_{Th(\nu)}\backslash -", "\simeq"'] \arrow[d, phantom, "\circlearrowright"] & C_{*+p}(Th(\nu)^*) \arrow[d, "(\Gamma_X)_*"] \\
C(X) \arrow[rr, "\Sigma^p", "\simeq"'] & & C_{*+p}(\Sigma^p X_+)
\end{tikzcd}$$

<u>Structures</u>

$$\begin{tikzcd}
U(\nu) \in H^k(Th(\nu)) \arrow[rr, "\alpha_{Th(\nu)}\backslash -"] \arrow[d, "-\cap h(\rho)"] & & H_{n+p}(Th(\nu))^* \arrow[dl, "(\Gamma_x)_*"'] \arrow[d, "\varphi"] \\
{[X] \in H_n(X)} \arrow[r] \arrow[d, "\varphi"] & H_{n+p}(\Sigma^p X) \arrow[d] & Q^{n+p}(C(Th(\nu)^*)) \arrow[dl, "\Gamma_x \otimes \Gamma_x"'] \arrow[d] \\
\varphi[X] \in Q^n(C(X)) \arrow[r] & Q^{n+p}(C(\Sigma^p X)) & \hat{Q}^{n+p}(C(Th(\nu)^*))
\end{tikzcd}$$

$\alpha_{Th(\nu)} \backslash U(\nu)$

$$\begin{tikzcd}
{} \arrow[d, mapsto] & {} \arrow[d, "f"] & {} \arrow[d] & {} \arrow[dl, "\widehat{\Gamma_X \otimes \Gamma_X}"'] & {} \arrow[d, "\cong"] & {} \\
f\circ\varphi[X] & \hat{Q}^n(C(X)) \arrow[r] & \hat{Q}^{n+p}(C(\Sigma^p X)) & & \hat{Q}^{n+p}(C^{N-*}(Th(\nu))) \arrow[d, "\cong"] \\
& & & & \hat{Q}^{n+p}(C^{n+p-*}(X)) \arrow[ull, "S^p(\varphi[X]_0)"'] \\
& & \hat{Q}^n(C^{n-*}(X)) \arrow[uul, "(\varphi[X])_0"] \arrow[urr] & & \\
& & & & \hat{Q}^0(C^{-*}(X)) \ni \gamma(\nu) \arrow[uu, "\cong"'] \arrow[ull, "S^n"]
\end{tikzcd}$$

"chain bundle"

Where we recall

$(\varphi[X])_0 = -\cap[X] : C^{n-*}(X) \to C(X)$

$S^p(\varphi[X])_0 = C^{n+p-*} \longrightarrow \Sigma^p C(X)$

So $\quad f\circ\varphi[X] \sim \# \hat{S}\gamma(\nu)$ (in partic)

<u>Def'n</u> Let $C \in \mathbb{B}(A)$

- A chain bundle on $C$ is a cycle $\gamma \in \operatorname{Hom}(\hat{W}, C^{-*} \otimes_A C^{-*})_0$

  alg normal cx.

- An n-dim ANC is a 4-tuple $(C, \varphi, \gamma, \chi)$
  - $C \in \mathbb{B}(A)$
  - $\varphi$ n-cycle in $\operatorname{Hom}(W, C \otimes_A C)$
  - $\gamma$ 0-cycle in $\operatorname{Hom}(\hat{W}, C^{-*} \otimes_A C^{-*})$
  - $\chi$ (n+1)-chain in $\operatorname{Hom}(\hat{W}, C \otimes_A C)$

s.t. $\quad \hat{\varphi}_0^{\%}(S^n(\gamma)) - f\gamma = d\chi \qquad$ (recall $\hat{\varphi}_0^{\%} = \widehat{\varphi_0 \otimes \varphi_0}$ )

~~Def'n/Prop X an n-GPC then we obtain~~

**FACTS** We get $NL^n(A)$ = normal bordism groups of ANCs

$$\Omega^N_n \longrightarrow \cancel{W}\, N\mathbb{L}^n(\mathbb{Z})$$

$$\exists\ \mathrm{sign}^{N\mathbb{L}^\bullet} : \Omega^N_\bullet \longrightarrow N\mathbb{L}^\bullet$$

**Def'n / Prop** $X$ $n$-dim GPC we obtain

$$\mathrm{sign}_X^{N\mathbb{L}^\bullet}(X) \in H_n(X; N\mathbb{L}^\bullet)$$

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## Lecture 14

Andrew Ranicki

$X$ a fin simplicial cx

$\Omega^P_n(X)$ = bordism of $n$-dim $n$-GPC $Q$ with map $g : Q \to X$

$\mathcal{S}_n(X) := \$_n(X)$ (Andrew uses a different font.)

Aim to define
$$s : \Omega^P_n(X) \longrightarrow \$_n(X)$$
$$(Q, g) \longmapsto g_* s(Q) \quad \text{where } s(Q) \in \$_n(Q)$$

Total surgery obstruction

We have a braid

$$\begin{tikzcd}
 & & H_n(X;\Omega^P) \arrow[dr, "A"] \arrow[rr, bend left] & & \Omega^N_n(X) \arrow[dr] & \\
 & H_n(X;\mathbb{L}_\bullet) \arrow[ur] \arrow[dr, "A"] & & \Omega^P_n(X) \arrow[ur] \arrow[dr] & & H_{n-1}(X;\mathbb{L}_\bullet) \\
\Omega^N_{n+1}(X) \arrow[ur] \arrow[rr, bend right, "\partial"] & & L_n(\mathbb{Z}\pi, X) \arrow[ur] \arrow[rr, bend right] & & \$_n(X) \arrow[ur] &
\end{tikzcd}$$

$$\partial : (Q^{n+1}, g) \longmapsto \partial(C(Q), \varphi_Q, \gamma_Q, \chi_Q) =: (D, \psi) \qquad (*)$$

$$\overset{MSO}{\downarrow}$$

- $\Omega_n^{SO}(X) := H_n(X; \Omega^{SO}(pt))$

- $\Omega_n^N(X) :=$ bordism of n-dim GNC $Q$ with map $g: Q \to X$
  $\cong H_n(X; \Omega^N(pt.))$
  $\uparrow$ MSG

(*) $D := \mathrm{cone}\left( (\varphi_Q)_0 : C(Q)^{n+1-*} \longrightarrow C(Q)\right)_{*+1}$

Why is $\Omega_n^P(X)$ not a generalised homology theory?
Because it doesn't have transversality. $\Omega^P$ builds this in:

- $H_n(X; \Omega_m^P(X)) = \left\{ \begin{array}{l} Q = n\text{-GPC}, g: Q \to X \text{ s.t. for each simplex } \sigma \in X \\ (g^{-1}D(\sigma), g^{-1}\partial D(\sigma)) \text{ an } (n-|\sigma|)\text{-GPP} \end{array} \right\}$

$$\Big\downarrow A \qquad \text{"forget fragmentation"}$$

$\Omega_n^P(X)$.

<u>Example</u> If $X$ is an $n$-GPC $\quad (X, 1 : X \to X) \in \Omega_n^P(X)$
then $s(X, 1 : X \to X) \in \mathbb{S}_n(X)$

<u>Theorem</u> (Main theorem) $n \geq 5$

$$s(X) = 0 \iff X \simeq \text{TOP mfld.}$$

## Another braid (now algebraic)

$$\begin{tikzcd}
& L^n(\mathbb{Z}_*(X)) \arrow[d, phantom, "\parallel"] \\
L_n(\mathbb{Z}_* X) \arrow[d, phantom, "\parallel"] & H_n(X;\mathbb{L}^\bullet) \arrow[dr, "A"] \arrow[rr, bend left] & & H_n(X; N\mathbb{L}^\bullet) \arrow[dr] \\
H_n(X;\mathbb{L}_\bullet) \arrow[ur, "1+T"] \arrow[dr, "A"] & & VL^n(\mathbb{Z}_*(X)) \arrow[ur] \arrow[dr, "S"] & & H_{n-1}(X;\mathbb{L}^\bullet) \\
& L_n(\mathbb{Z}\pi, X) \arrow[ur] \arrow[rr, bend right] & & \mathbb{S}_n(X) \arrow[ur, "\partial"]
\end{tikzcd}$$

$VL^n(\mathbb{Z}_*(X))$ = "visible L-groups of M. Weiss"

$\Downarrow$

$V\mathrm{sign}^L(X)$ = "visible signature"

## Visible L-groups

**Recall** $C \otimes_{\mathbb{Z}\pi} C = C \otimes_{\mathbb{Z}} C \big/ \{gx \otimes y - y \otimes gx\}$

and $T(x \otimes y) = \pm y \otimes x$

and $\mathbb{Z} \otimes_{\mathbb{Z}\pi} C(\tilde{X}) = C(X) \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}\pi} \left( C(\tilde{X}) \otimes_{\mathbb{Z}} C(\tilde{X}) \right)$

$= C(\tilde{X}) \otimes_{\mathbb{Z}\pi} C(\tilde{X})$

~~Brilliant idea~~:

## Brilliant idea

Let $P: \quad \ldots \to P_2 \to P_1 \to P_0$ be a free $\mathbb{Z}\pi$-module resolution of $\mathbb{Z}$ with trivial $\pi$-action. e.g. take $P = C(E\pi)$ $\left(\begin{smallmatrix} E\pi \\ \downarrow \\ B\pi \end{smallmatrix}\right)$

Instead of applying $\mathbb{Z} \otimes_{\mathbb{Z}\pi}$ apply $P \otimes_{\mathbb{Z}\pi}$

$$\Rightarrow \quad \begin{array}{c} P \otimes_{\mathbb{Z}\pi} C(\tilde{X}) \\ \wr\!\wr \\ C(X) \end{array} \xrightarrow{1 \otimes \tilde{\Delta}_0} \underbrace{P \otimes_{\mathbb{Z}\pi} (C(\tilde{X}) \otimes_{\mathbb{Z}} C(\tilde{X}))}_{=: V}$$

$$\Rightarrow H_n(X) \xrightarrow{V\varphi} H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, P \otimes (C(\tilde{X}) \otimes_{\mathbb{Z}} C(\tilde{X}))$$

$$=: VQ^n(C(\tilde{X}))$$

So we have

$$\begin{tikzcd} H_n(X) \arrow[r, "V\varphi"] \arrow[dr, "\varphi"'] & VQ^n(C(\tilde{X})) \arrow[d] \\ & Q^n(C(\tilde{X})) \end{tikzcd}$$

<u>Def'n</u> A symmetric form $(a_{ij})$ is visible if $a_{ii} \in \mathbb{Z} \subseteq \mathbb{Z}\pi / \{x + \bar{x}\}$

<u>Example</u> $\pi = \mathbb{Z}_2$ $\quad \mathbb{Z}\pi = \mathbb{Z}[t]/t^2-1$

The symmetric form $(\mathbb{Z}\pi, t)$ is not visible.

$$\Omega_{4k}(B\mathbb{Z}_2) \xrightarrow{\mathrm{sign}^L} L^{4k}(\mathbb{Z}[\mathbb{Z}_2])$$

IS NOT IN THE IMAGE.

**Def'n** $VL^n(\mathbb{Z}_*(X))$ = cobordism of chain cx $C$ in $\mathbb{Z}_*(X)$ with visible symm $\varphi$ Poincaré structure such that

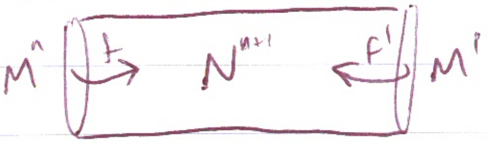
$$\varphi \in Q^n_{\mathbb{Z}_*X}(C) = H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_{\mathbb{Z}_*X} C))$$

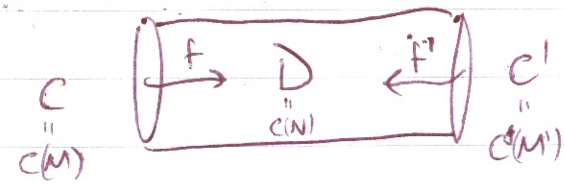
and $\varphi_0 : C(\tilde{X})^{n-*} \to C(\tilde{X})$ an iso.

Remark
- If $X = B\pi$ $\quad Q^n_{\mathbb{Z}_*B\pi}(C) = VQ^n(A(C))$.
- If $X = n$-GPC $\quad \mathrm{sign}^{VL}(X) = (C(X'), \varphi) \in \mathbb{B}(\mathbb{Z}_*X)$.

# Lecture 15 — Alg vs. Geom Surgery

Andrew Ranicki




**Poincaré–Lefschetz**

$$H^{n+1-*}(N,M') \cong H_*(N,M)$$

$$(\delta\varphi, \varphi \oplus -\varphi) \in Q^{n+1}((f\ f') : C \oplus C' \to D)$$

$$\begin{pmatrix} \delta\varphi_0 & - \\ - & - \end{pmatrix} : \operatorname{cone}(f')^{n+1-*} \longrightarrow \operatorname{cone}(f)$$

is a chain equiv

To pass between the two we use the symmetric construction

$$\varphi_M : H_n(M) \longrightarrow Q^n(C(M)) \quad \text{for } M$$

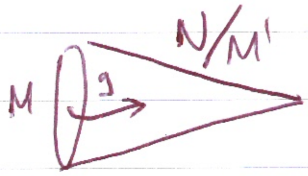
$$\varphi_{(N;M,M')} : H_{n+1}(N, \partial N) \longrightarrow Q^{n+1}((f\ f') : C \oplus C' \to D)$$

**Theorem** (version 1) Can recover the $(n+1)$-dim SAP-cobordism $(f\ f')$ from the $(n+1)$-dim symmetric pair

$$(g : C \longrightarrow \underset{= D/C'}{\operatorname{cone}(f')}, (\delta\varphi/\varphi', \varphi))$$

NOTE: WITHOUT POINCARÉ DUALITY!

**Idea:**




collapse $M'$

$$\Longrightarrow \quad C \to D \to \operatorname{cone}(f'), \quad g : C \to \operatorname{cone}(f')$$

$$\Rightarrow Q^{n+1}(g) \to Q^n(C) \xrightarrow{g^{\%}} Q^n(\mathrm{cone}(f'))$$

Where

<u>Def'n</u> $g : C \to \overset{\mathrm{cone}(f')}{\overset{\shortparallel}{E}} \Rightarrow g \otimes g : C \otimes C \to E \otimes E$
$\Rightarrow g^{\%} : Q^n(C) \to Q^n(E)$

<u>Theorem</u> (version 2) There is a 1:1 correspondence between homotopy equivalences classes of $(n+1)$-dim symm Poincaré cobordisms and $(n+1)$-dim Poincaré pairs with $(C,\varphi)$ PC.

$$\left\{((f\ f') : C \oplus C' \to D, (\delta\varphi, \varphi \oplus \varphi'))\right\}/\sim \; \longleftrightarrow \; \left\{(g : C \to E, (\delta\varphi', \varphi))\right\}/\sim$$

<u>Proof</u> (Method)

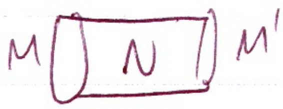
$(\Rightarrow)$ $(f\ f') \rightsquigarrow (g : C \to D \hookrightarrow E = \mathrm{cone}(f'))$

$(\Leftarrow)$ Requires algebraic surgery.

$(g : C \to E, (\delta\varphi', \varphi))$ is the input for <u>algebraic surgery</u>.

Geometrically:

Suppose $M'$ is the effect of ~~a~~ 1 $k$-surgery




$N = (M \times I) \cup (D^{k+1} \times D^{n-k})$

$\Rightarrow M' = \overline{M \setminus S^k \times D^{n-k}} \cup (D^{k+1} \times S^{n-(k+1)})$

WARNING! CAN'T GET $M'$ UNIQUELY.

We can recover $C(M')$ from $\overset{\text{symmetric pair}}{\left(C(M) \longrightarrow C(N,M')\right)}$, ~~dn~~

<u>Claim</u> $C(M')_r \simeq C(M)_r \oplus C(N,M')_{r+1} \oplus C(N,M')^{n-r+1}$

with $d_{C(M')} = \begin{pmatrix} d_{C(M)} & 0 & S\varphi_0 \\ g & d_{C(N,M')} & \varphi_0 g^* \\ 0 & 0 & d_{C(N,M)^*} \end{pmatrix}$ $\square$

Algebraically: $D = \mathrm{cone}(g : C \to E)$
$C' = \mathrm{cone}\left(\begin{pmatrix} S\varphi_0 \\ \varphi_0 g^* \end{pmatrix} : E^{n+1-*} \longrightarrow D\right)$

with differentials:

$$\begin{array}{ccccc}
C_r & \longrightarrow & D_r = E_r \oplus E_{r+1} & \longleftarrow & C'_r = C_r \oplus E_{r+1} \oplus E^{n-r+1} \\
\downarrow d_C & & d_D \downarrow \begin{pmatrix} d_C & 0 \\ g & d_E \end{pmatrix} & & \begin{pmatrix} d_C & 0 & S\varphi_0 \\ g & d_E & \varphi_0 g^* \\ 0 & 0 & d_{E^*} \end{pmatrix} \downarrow d_{C'} \\
C_{r-1} & \longrightarrow & D_{r-1} = C_{r-1} \oplus E_r & \longleftarrow & C'_{r-1} = C_{r-1} \oplus E_r \oplus E^{n-r+2}
\end{array}$$

$\square$

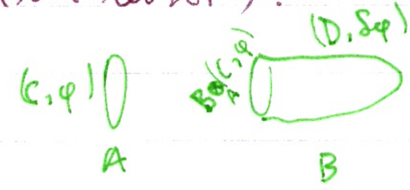
<u>Special cases</u>: $\boxed{C = 0}$

$\left\{ \begin{array}{l} (n+1)\text{-dim symm Poincaré} \\ \text{pairs } (C' \xrightarrow{f'} D, (S\varphi, \varphi')) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} (n+1)\text{-dim sym complexes} \\ (E = \mathrm{cone}(f'), S\varphi/\varphi') \end{array} \right\}$

# ~~Localisation~~ Relative L-groups

A morphism of rings with involution $f: A \to B$ induces a morphism $f: L_n(A) \to L_n(B)$ by $(C,\varphi) \mapsto B \otimes_A (C,\varphi)$ (using the $(B,A)$-bimodule structure to form the tensor).

Want $L_n(f)$ to fit into:




$$\cdots \to L_n(A) \to L_n(B) \to L_n(f) \to L_{n-1}(A) \to \cdots$$

with $L_n(f) := \left\{ \overset{\text{rel}}{\text{cobordism}} \text{ group of pairs } \left((C,\varphi), (g: B\otimes_A C \to D, (\delta\varphi, 1\otimes\varphi))\right)\right\}$

Where $(C,\varphi)$ - $(n-1)$-dim quad PC over $A$

$(g: B\otimes_A C \to D, (\delta\varphi, 1\otimes\varphi))$ $n$-dim quad PP over $B$

and $L_n(B) \to L_n(f)$; $(D, \delta\varphi) \mapsto (0, (0: 0 \to D\,(\delta\varphi, 0)))$
$L_n(f) \to L_{n-1}(A)$; $(\sim\!\sim) \mapsto (C,\varphi)$

---

Going back to symmetric ...

<u>Def'n</u> The <u>boundary</u> of an $(n+1)$-dim symmetric complex $(E,\theta)$ is the $n$-dim symm Poincaré cx $(\partial E, \partial\theta)$ with (*)

$$\partial E = \operatorname{cone}(\theta_0 : E^{n+1-*} \to E)_{*+1}$$

$$d_{\partial E} = \begin{pmatrix} d & \theta_0 \\ 0 & d^* \end{pmatrix}$$

with $\partial\theta$ defined as:

$$\partial\theta = \begin{pmatrix} \theta & 1 \\ 1 & 0 \end{pmatrix} : \underset{\overset{\|}{\partial E^{n-r}}}{E^{n-r+1} \oplus E_{n+1}} \longrightarrow \underset{\overset{\|}{(\partial E^{n-r})^* = \partial E_r}}{E_{r+1} \oplus E^{n-r+1}}$$

Consequence: an $n$-dim symm Poincaré complex $(C,\varphi)$ over $A$ is s.t. $(C,\varphi) = 0 \in L^n(A)$ if and only if $(C,\varphi)$ is htpy equiv to the boundary $\partial(E,\theta)$ of an $(n+1)$-dim symm cx $(E,\theta)$.

## Localisation exact sequence

Suppose now that $f: A \longrightarrow B = S^{-1}A$ is the inclusion ~~of~~ with $B$ the localisation inverting a subset of central non-zero divisors $S$ multiplicatively closed $1 \in S$.

$$\Rightarrow B = S^{-1}A = \left\{ a/s \;\middle|\; \frac{a}{s} = \frac{at}{st} \quad \forall t \in A \right\}$$

Claim $L^n(A \longrightarrow S^{-1}A) = L^n(A,S)$ and $L^n(S^{-1}A) \to L^n(A,S)$ is the boundary (*)

Claim Clearing fractions gives

$$L^n(S^{-1}A) = \left\{ \begin{array}{l} \text{cobordism group of } n\text{-dim symmetric} \\ S^{-1}A\text{-Poincaré complexes } (C,\varphi) \text{ over } A \end{array} \right\}$$

$H_*(S^{-1}A \otimes_A \mathrm{cone}(\varphi_0)) = 0$

Def'n $$L^n(A,S) = \left\{ \begin{array}{l} (n-1)\text{-dim symm Poincaré } S^{-1}A\text{-contractible} \\ \text{complexes over } A \ (C,\varphi) \end{array} \right\}$$

i.e.

$H_*(S^{-1}A \otimes_A C) = 0$

ecture 16

# Quadratic ~~vs~~ and Symmetric vs. Normal Complexes

Tibor Macko

Aim

Use $\partial$ to explain the relation between $L_n, L^n, NL^n$ for $\mathbb{A}$ an additive category with chain duality.

Recall

For a fixed $C \in B(\mathbb{A})$ we have

$$\cdots \to Q_n(C) \xrightarrow{1+T} Q^n(C) \xrightarrow{f} \hat{Q}^n(C) \xrightarrow{H} Q_{n-1}(C) \to \cdots$$

$$= \cdots \to H_n(W \otimes_{\mathbb{A}} (C \otimes_{\mathbb{A}} C)) \xrightarrow{1+T} H_n(\mathrm{Hom}_{\mathbb{Z}_2}(W, C \otimes_{\mathbb{A}} C)) \xrightarrow{f} H_n(\mathrm{Hom}(\hat{W}, C \otimes_{\mathbb{A}} C)) \xrightarrow{H} \cdots$$

and recall $\underset{\overrightarrow{k}}{\mathrm{colim}}\, Q^{n+k}(S^k C) = \hat{Q}^n(C)$.

Last lecture: ~~A~~

$$L_n(\mathbb{A}) \xrightarrow{1+T} L^n(\mathbb{A}) \longrightarrow \begin{pmatrix}\text{relative} \\ \text{term}\end{pmatrix}_n \longrightarrow L_{n-1}(\mathbb{A})$$

$$(C, \psi) \longmapsto (C, (1+T)\psi)$$

$$(C, \varphi) \overset{\partial}{\longmapsto} ???$$

and relative terms should be $(C \xrightarrow{f} D, (\delta\varphi, (1+T)\psi))$

Recall boundary:

$(C, \varphi)$ n-dim symm cx (non-Poincaré)

$$\varphi_0 : C^{n-*} \longrightarrow C_*$$

$\Rightarrow (\partial C, \partial\varphi)$ with $\partial C := \Sigma^{-1} \operatorname{cone}(\varphi_0)$

(so there is a cofibration: $\partial C \to C^{n-*} \xrightarrow{\varphi_0} C \xrightarrow{e} \operatorname{cone}(\varphi_0)$ )

$e$ induces $e^{\%} : Q^n(C) \longrightarrow Q^n(\operatorname{cone}(\varphi_0))$ and we want to take $\partial\varphi$ as Andrew defined it. We claim there is the property:

$\boxed{\text{EXERCISE}}$

$$\begin{array}{ccc} Q^{n-1}(\partial C) & \xrightarrow{S} & Q^n(S\partial C) \simeq Q^n(\operatorname{cone}(\varphi_0)) \\ \partial\varphi & \longmapsto & e^{\%}(\varphi) \end{array}$$

---

<u>n-ANC</u> $(C, \varphi, \gamma, \chi)$ ~~with~~ What does the data mean?

$C \in \mathbb{B}(A)$

~~$\varphi \in Q^n(C)$~~

cycle – $\varphi \in Hom(W, C \otimes C)$

cycle – $\gamma \in Hom(\hat{W}, C \otimes C)$

not a cycle – $\chi \in Hom(W, C \otimes C)_{n+1}$

$$\begin{array}{ccccccc} \hat{Q}^0(C^{n-*}) & \xrightarrow{\cong} & \hat{Q}^n(C^{-*}) & \xrightarrow{\hat{\varphi}_0^{\%}} & \hat{Q}^n(C) & \xleftarrow{f} & Q^n(C) \\ \gamma & \longmapsto & S^n(\gamma) & \longmapsto & \text{compare} & \longleftarrow & \varphi \end{array}$$

but $d\chi = \hat{\varphi}_0^{\%}(S^n(\gamma) - f(\varphi_0))$

**Prop** $(C,\varphi)$ $n$-SAC in $\mathbb{A}$ (not Poincaré)

$\exists\, \gamma \in \hat{Q}^0(C^{-*})$ s.t. $\hat{\varphi}_0^{\%}(S^n(\gamma)) = f(\varphi)$

$\Longleftrightarrow$ $\exists\, \partial\psi\, Q_{n-1}(C)$ s.t. $(1+T)\partial\psi = \partial\varphi$.

**Proof**

$$
\begin{tikzcd}
 & \overset{\partial\psi}{Q_{n-1}(\partial C)} \arrow[rr, "1+T"] & & \overset{\partial\varphi}{Q^{n-1}(\partial C)} \arrow[rr, "f"] \arrow[dr, "S"] & & \hat{Q}^{n-1}(\partial C) \ni f(\partial\varphi) \arrow[dd, bend left, "S", "\cong"'] \\
 & & \varphi \in Q^n(C) \arrow[rr, "e^{\%}"] \arrow[dr, "f"] & & \overset{e^{\%}(\varphi)}{Q^n(\mathrm{cone}(\varphi_0))} \arrow[dr, "f"] & \\
 & \underset{S^n(\gamma)}{\hat{Q}^n(C^{n-*})} \arrow[rr, "\hat{\varphi}_0^{\%}"] & & \underset{f(\varphi)}{\hat{Q}^n(C)} \arrow[rr, "\hat{e}^{\%}"] & & \underset{\hat{e}^{\%}(f(\varphi))}{\hat{Q}^n(\mathrm{cone}(\hat{\varphi}_0))} \\
\underset{\gamma}{\hat{Q}^0(C^{-*})} \arrow[ur, "\cong"'] & & & & &
\end{tikzcd}
$$

So $f(\partial\varphi) = 0 \Longleftrightarrow \hat{e}^{\%}(f(\varphi)) = 0$

So $\partial\varphi$ lifts $\Longleftrightarrow$ $f(\varphi)$ lifts

$\Rightarrow$ proposition as we now have $\gamma, \partial\psi$ $\square$

---

Andrew Ranicki

**Recall** The formula for the quadratic boundary of an $n$-dimensional normal complex $(C, \varphi, \gamma, \chi)$

up to sign

$\varphi \in Q^n(C)$ $\quad \varphi_s : C^{n-r+s} \longrightarrow C_r$ $\quad d\varphi_s + \varphi_s d^* + \varphi_{s-1} + \varphi_{s-1}^* = 0$

$\gamma \in \hat{Q}^0(C^{-*})$ $\quad \gamma_s : C_r \longrightarrow C^{-r-s}$ $\quad d^*\gamma_s + \gamma_s d + \gamma_{s-1} + \gamma_{s-1}^* = 0$

$\chi$ $\quad \chi_s : C^{n-r+s} \longrightarrow C_{r+1}$

M. Weiss calls this the "labelling function"

$$\varphi_0 \gamma_{s-n} \varphi_0^* - \varphi_s = d\chi_s + \chi_s d^* + \chi_{s-1} + \chi_{s-1}^* \quad (*)$$

$: C^{n-r+s} \longrightarrow C_r$

$$\partial(C,\varphi,\gamma,\chi) = \Big(\big(\mathrm{cone}(\varphi_0 : C^{n-*} \to C)_{*+1}, \psi\big)$$

$$\overset{\partial C}{\shortparallel}$$

$$\psi_s = \begin{pmatrix} \chi & \text{sthg} \\ \text{sthg} & \text{sthg} \end{pmatrix} : \partial C^{n-r-s} \longrightarrow \partial C_r$$

the appearance of $\psi$ is the important thing here.

$$NL^n(\mathbb{A}) \longrightarrow L_{n-1}(\mathbb{A})$$

<u>Question</u> How does a n-dim GNC determine an n-dim ANC in $\mathbb{Z}_* X$?

$$(X,\nu,\rho) \longmapsto \Big(\big(C(X), \varphi([X])\big), \gamma(\nu), \chi(\rho)\Big)$$

- It is clear that $[X] = \rho_*(S^{n+k}) \in H_{n+k}(Th(\nu)) \cong H_n(X)$ should be $[X]$.

- $\gamma(\nu)$ should be $H^k(Th(\nu)) \xrightarrow{\cong} H_n(Th(\nu)^*) \xrightarrow{\varphi_{Th}} \hat{Q}^n(C(X)^{n-*}) \xrightarrow{\cong} \hat{Q}^0(C(X)^{-*})$

$$U_\nu \longmapsto \gamma(\nu).$$

- $\chi$ is very delicate and captures information like the Kervaire invariant:

For any space $T$ there is a LES coming from the Hurewicz map $h$

$$\cdots \to \Gamma_n(T) \to \pi_n(T) \xrightarrow{h} H_n(T) \to \Gamma_{n-1}(T) \to \cdots$$

$\shortparallel$

"quadratic structures"

cf. Whitehead "A certain exact sequence".

We have

$$\begin{tikzcd}
{} \arrow[r] & \Gamma_{n+k}(Th(\nu)) \arrow[r] & \pi_{n+k}(Th(\nu)) \arrow[r, "h"] \arrow[dd] & H_{n+k}(Th(\nu)) \arrow[r] \arrow[d, "\mathrm{Thom}\ \cong"'] & \Gamma_{n+k-1}(Th(\nu)) \arrow[dd, "\theta_{Th(\nu)}"] \\
& & & H_n(X) \arrow[d, "\varphi_X"] & \\
& & Q_n(C(X), \gamma(\nu)) \arrow[r] & Q^n(C(X)) \arrow[r, "J_{\gamma(\nu)}"] & \hat{Q}^n(C(X))
\end{tikzcd}$$

$Q_n(C(X),\gamma(\nu))$ $=$ abelian group of all $(\varphi, \chi)$ satisfying $\circledast$

(And in fact we can consider this all for the additive category with chain duality $\mathbb{Z}_*(X)$.)

The image of $(\theta, \varphi)(\rho)$ under

$$Q_n^{\mathbb{Z}_*X}(C(X'), \gamma(\nu)) \longrightarrow Q^n_{\mathbb{Z}_*X}(C(X'))$$

is $(C(X'), \varphi(X), \gamma(\nu), \chi(\rho))$, an $n$-dim ANC in $\mathbb{Z}_*X$ with boundary

$$\partial(C, \varphi, \gamma, \chi) = (\partial C, \psi) \text{ an } (n-1)\text{-dim AQP in } \mathbb{Z}_*X$$

with assembly $A(\partial C) = \mathrm{cone}(-\cap[X] : C(\tilde{X})^{n-*} \longrightarrow C(\tilde{X}))$ in $\mathbb{Z}\pi_1 X$, which represents the failure of $(C, \varphi, \gamma, \chi)$ to be Poincaré.

can now describe this

Recall the surgery exact sequence braid with visible L-groups from lecture 14:

$$VL^n(\mathbb{Z}_*X) \xrightarrow{\partial} \mathbb{S}_n(X)$$

$VL^n(\mathbb{Z}_*X)$ = n-dim ASC in $\mathbb{Z}_*X$ $(C, \varphi, \gamma, X)$ which are $\mathbb{Z}\pi_1 X$-Poincaré

$\mathbb{S}_n(X)$ = $(n-1)$-dim QPC $(B, X)$ s.t. assembly$(B) = \mathbb{Z}\pi_1$ contractible

$TSO(X) = \partial(C(X), \varphi(X), \gamma(\nu), \rho(\nu))$
$= (B, \psi)$

$TSO(X) = 0 \in \mathbb{S}_n(X)$
$\Longleftrightarrow$ $X$ has an $\mathbb{L}^\bullet$-orientation up to alg cob (see braid)
$\Longleftrightarrow$ $X \simeq$ TOP mfld for $n \geq 5$

cf. Localisation

Lecture 17 — Tibor Macko

**Recall** Q: If $X$ n-GPC, $\exists$ n-mfld $M \simeq X$?

$$\mathbb{S}_n(X) := L_{n-1}\left(\begin{array}{l}\text{(n-1)-dim quadratic chain complexes} \\ \text{in } \mathbb{Z}_*X \text{ s.t. locally Poincaré + globally contractible}\end{array}\right)$$

Locally Poincaré – We are over simplices so we have $\varphi_0$ for lots of "fragments". We want each of these to be a chain htpy equiv. Maybe not true after assembly.

Globally contractible – Possible to have non-contractible ~~global~~ local info that assembles to something contractible.

$$\boxed{TSO(X) = s(X) = \partial\, \mathrm{sign}^{VL^\bullet}(X)} \quad \text{Total surgery obstruction}$$

where $\mathrm{sign}^{VL^\bullet}(X) \in VL^n(\mathbb{Z}_*X) = \left\{\begin{array}{l}\text{cobordism group of n-dim} \\ \text{normal complex in } \mathbb{Z}_*X \text{ globally Poincaré}\end{array}\right\} / \sim$

Elements $(C, \varphi, \gamma, \chi) \in VL^n(\mathbb{Z}_*X)$ are

$C(X') = C \in \mathbb{B}(\mathbb{Z}_*X)$ i.e. $\forall \sigma \in X \quad C(\sigma) = C(\underbrace{D(\sigma, X')}_{\text{dual cell}}, \partial D(\sigma, X'))$

+ $\varphi, \gamma, \chi$ defined algebraically in previous lectures.

Recall as well that:

$$\mathrm{sign}_X^{NL^\bullet}(X) \in H_n(X; NL^\bullet) = NL^n(\mathbb{Z}_*X)$$

$$= \left\{\begin{array}{l}\text{cobordism group of n-dim} \\ \text{normal complex in } \mathbb{Z}_*X\end{array}\right\} / \sim$$

forgetful map

Recall as well that:

$$(\varphi_X[X])_0 : C^{n-*}(X') \longrightarrow C(X) \qquad \text{over } \mathbb{Z}_* X$$

$$(\varphi_X[X])_0(\sigma) : C^{n-|\sigma|-*}(D(\sigma)) \longrightarrow C(D(\sigma), \partial D(\sigma))$$

so $s(X) = [D, \Psi]$ with $D = \partial C(X')$ in $\mathbb{Z}_* X$

$$D(\sigma) = \Sigma^{-1}\mathrm{cone}((\varphi_X[X])_0(\sigma))$$

**Remark** If $X$ is already a triangulated manifold then

$(\varphi_X[X])_0(\sigma)$ is always a htpy equiv $\forall \sigma$
$D(\sigma) \simeq *$ $\forall \sigma$ $\Big\}$ so we're in good shape.

The main point is the converse.

## MAIN THEOREM

$$0 = s(X) \in \mathbb{S}_n(X) \Leftrightarrow \exists M^n \text{ n-mfld s.t. } M^n \simeq X^n$$

## MAIN DIAGRAM

$$\begin{tikzcd}
 & H_n(X;\mathbb{L}^\bullet\langle 0\rangle) \arrow[rr, bend left] \arrow[dr, "A"] & & H_n(X;N\mathbb{L}^\bullet\langle \tfrac12\rangle) \arrow[rr, bend left, "\partial"] \arrow[dr, "\partial"] & & L_{n-1}(\mathbb{Z}\pi) \\
H_n(X;\mathbb{L}_\bullet\langle 1\rangle) \arrow[ur] \arrow[dr, "A"] & & VL^n(\mathbb{Z}_* X) \arrow[ur, "\text{forget}"] \arrow[dr, "\partial"] & & \underset{t(X)}{H_{n-1}(X;\mathbb{L}_\bullet\langle 1\rangle)} \arrow[ur] & \\
 & L_n(\mathbb{Z}\pi) \arrow[ur] \arrow[rr, bend right, "\partial"] & & \underset{s(X)}{\mathbb{S}_n(X)} \arrow[ur] & &
\end{tikzcd}$$

# Orientations

**Setup** $\alpha : X \to BSG(k)$

$\mathbb{E}$ a ring spectrum / $\Omega$-spectrum $\quad pr : \mathbb{E} \times \mathbb{E} \to \mathbb{E}$
unit $\mathbb{S}^0 \to \mathbb{E}$

$\mathbb{E}^*(S^k)$ free $\pi_*(\mathbb{E})$-module on one generator

**Def'n** A $\mathbb{E}$-orientation of $\alpha$ is a stable htpy class of maps $u^{\mathbb{E}}(\alpha) : Th(\alpha) \to \mathbb{E}_k$ s.t. $u^{\mathbb{E}}(\alpha)|_{Th(\alpha|_x)} : S^k \to \mathbb{E}_k$ is a generator $\forall x \in X$.

**Prop/Def** (Canonical orientation)
(1) $\forall \alpha : X \to BSG(k) \; \exists \, u^{MSG}(\alpha) : Th(\alpha) \to MSG$
MSG built from $Th(VSG)$.
(2) $\forall \beta : X \to BSTOP(k) \; \exists \, u^{MSTOP}(\beta) : Th(\beta) \to MSTOP$
(3) $u^{MSG}(J(\beta)) \simeq J u^{MSTOP}(\beta)$.

---

$\Omega^N_\bullet$, $\Omega^{STOP}_\bullet$ were constructed with N-ads

**FACT** (Pont/Thom construction) $\Omega^{STOP}_\bullet \simeq MSTOP$, $\quad \Omega^N_\bullet \simeq MSG$

**Recall**
$\mathrm{sign}^{NL^\bullet} : \Omega^N_\bullet \to NL^\bullet\langle 1/2 \rangle$
$\mathrm{sign}^{\mathbb{L}^\bullet} : \Omega^{STOP}_\bullet \to \mathbb{L}^\bullet\langle 0 \rangle$
were maps of ring spectra.

We have not proved any of these sequences are exact.

① ASES

Need to use loc. ~~techniques~~ + π-π theorem

$$H_n(X;\mathbb{L}_\bullet) \longrightarrow L_n(\mathbb{Z}\pi) \longrightarrow \mathbb{S}_n(X)$$

② $$L_n(\mathbb{Z}\pi) \longrightarrow VL^n(\mathbb{Z}_*X) \longrightarrow H_n(X;N\mathbb{L}^\bullet)$$ " ___ "

③ $$H_n(X;\mathbb{L}_\bullet) \longrightarrow H_n(X;\mathbb{L}^\bullet) \longrightarrow H_n(X;N\mathbb{L}^\bullet)$$

this is a spectrum version of $L_n \to L^n \to NL^n$

PROOF OF THEOREM Definitely need to fold in geom surgery! Manifolds don't spring from algebra alone.

(⟹) We show (i) $\underset{\overset{\shortparallel}{0}}{t(X)} \in H_{n-1}(X;\mathbb{L}_\bullet\langle 1\rangle)$ iff

$$\begin{tikzcd} & BSTOP \arrow[d] \\ X \arrow[ur, dashed, "\nu"] \arrow[r, "\nu_X"'] & BSG \end{tikzcd}$$

there is a lift

(ii) If $t(X)=0$ then

$$L_n(\mathbb{Z}\pi) \supset \partial^{-1}(s(X)) = \{\operatorname{sign}^{\mathbb{L}_\bullet}_{\mathbb{Z}\pi}(f,b) \mid (f,b): M \xrightarrow{\deg 1} X,\ M \text{ n-mfld}\}$$

(i) Want to show that $\partial^{-1}t(X) = \operatorname{sign}^{N\mathbb{L}^\bullet}_X(X)$ has a preimage in $H_n(X;\mathbb{L}^\bullet\langle 0\rangle)$ (sequence ③).

**Prop/Def'n** (1) $\forall \alpha : X \to BSG(k)\ \exists\, u^{N\mathbb{L}^\bullet}(\alpha) : Th(\alpha) \to N\mathbb{L}^\bullet\langle \tfrac{1}{2}\rangle$

(2) $\forall \beta : X \to BSTOP(k)\ \exists\, u^{\mathbb{L}^\bullet}(\beta) : Th(\beta) \to \mathbb{L}^\bullet\langle 0\rangle$

(3) $u^{N\mathbb{L}^\bullet}(J(\beta)) \simeq J u^{\mathbb{L}^\bullet}(\beta)$

**Proof** Induced from maps of ring spectra. $\square$

In fact $u^{N\mathbb{L}^\bullet}(\nu_X) \in H^k(Th(\nu_X); N\mathbb{L}^\bullet\langle \tfrac{1}{2}\rangle) \underset{\text{S-duality}}{\cong} H_n(X; N\mathbb{L}^\bullet\langle \tfrac{1}{2}\rangle)$

$\beta = \gamma$ $\quad u^{\mathbb{L}^\bullet}(\nu_X) \in H^k(Th(\nu_X); \mathbb{L}^\bullet\langle 0\rangle) \cong H_n(X; \mathbb{L}^\bullet\langle 0\rangle)$

**Prop** $\alpha_X \backslash u^{N\mathbb{L}^\bullet}(\nu_X) = \mathrm{sign}_X^{N\mathbb{L}^\bullet}(X)$

$\alpha_X \backslash u^{\mathbb{L}^\bullet}(\nu) = \mathrm{sign}_X^{\mathbb{L}^\bullet}(M)$ $\quad$ (for $(f,b) : M \xrightarrow[\text{normal}]{\deg 1} X$)

---

~~We now know that reduction of $\nu_X$ to TOP $\Rightarrow$~~

**Setup** $E$ ring spec $\pi_0(E) = \mathbb{Z}$ induced $E \xrightarrow{h} H\mathbb{Z}$ (Eilenberg-Maclane spectrum)
$E_\otimes$ = component of 1

[MAY]: $\exists$ classifying space $BEG$ for $H\mathbb{Z}$-oriented ~~spaces~~ spherical fibrations with $E$-orientation compatible with the $H\mathbb{Z}$-structure.

**Th'm** 1:1 corr

$$[X, BEG] \longleftrightarrow \left\{ \begin{array}{l} \alpha : X \to BSG(h) \\ u^{\mathbb{E}}(\alpha) : Th(\alpha) \to \mathbb{E} \\ \text{s.t. } h(u^{\mathbb{E}}(\alpha)) = u^{H\mathbb{Z}}(\alpha) \end{array} \right\} \Big/ \sim$$

**Theorem** $\exists$ fib seq

$$\mathbb{E}_{\otimes} \to BEG \to BSG$$

**Theorem** $\exists$ htpy pullback diagram

$$\begin{tikzcd} BSTOP \arrow[r, "u^{\mathbb{L}^\bullet}"] \arrow[d] & B\mathbb{L}^\bullet\langle 0\rangle G \arrow[d] \\ BSG \arrow[r, "u^{N\mathbb{L}^\bullet}"] & BN\mathbb{L}^\bullet\langle \tfrac{1}{2}\rangle G \end{tikzcd}$$

take canonical orientation

**Remark** Enough now to obtain lift of ~~...~~ by htpy pullback.

$$\begin{tikzcd} & & B\mathbb{L}^\bullet\langle 0\rangle G \arrow[d] \\ X \arrow[r] \arrow[urr, dashed] & BSG \arrow[r] & BN\mathbb{L}^\bullet\langle \tfrac{1}{2}\rangle G \end{tikzcd}$$

**Proof** We show the two htpy fibres are the same.

$$\begin{tikzcd} G/TOP \arrow[r, "\text{This is htpy..}"] \arrow[d] & \mathbb{L}_0\langle 1\rangle \arrow[d] \\ BSTOP \arrow[r] \arrow[d] & B\mathbb{L}^\bullet\langle 0\rangle G \arrow[d] \\ BSG \arrow[r] & BN\mathbb{L}^\bullet\langle \tfrac{1}{2}\rangle G \end{tikzcd}$$

$\square$.

(ii) Remark on this part of the proof.

$X$ a manifold:

$$
\begin{tikzcd}
\mathcal{N}^{TOP}(X) \arrow[r, "\mathrm{sign}^{\mathbb{L}^\cdot}_{\mathbb{Z}\pi} - \mathrm{sign}^{\mathbb{L}^\cdot}_{\mathbb{Z}\pi}(f_0, b_0)"] \arrow[d, "\mathrm{sign}^{\mathbb{L}^\cdot}_X - \mathrm{sign}^{\mathbb{L}^\cdot}_X(f_0, b_0)"] & L_n(\mathbb{Z}\pi) \arrow[d, "="] \\
H_n(X; \mathbb{L}_\cdot\langle 1 \rangle) \arrow[r, "A"'] & L_n(\mathbb{Z}\pi)
\end{tikzcd}
$$

commutes.

Furthermore we now have an ~~ex~~ identification between the alg and geo surgery exact sequences.

"$\square$"