Seiberg Witten and homotopy theory

Lecture 1

Recently Manolescu resolved the triangulation conjecture

Goal Explain the ideas that go into this.

The SW equations

$$D_A\phi = 0, \quad \tfrac{1}{2}F_{A^t}^+ + \rho^{-1}(\phi\phi^*)_0 = 0$$

The classical approach is to study solutions.

Bauer, Furuta, Manolescu study the equations themselves, perhaps making it clearer where the topology appears.

Spin and spin^c structures

Today: X closed oriented Riemannian 4-mfld.

$$\mathrm{Spin}^c(4) \cong \{(a_+, a_-) \in U(2)\times U(2) \mid \det a_+ = \det a_-\}$$

$$\cup$$

$$\mathrm{Spin}(4) \cong \quad SU(2)\times SU(2)$$

$$\mathbb{H} = \{a + bI + cJ + dK\} = \left\{\begin{pmatrix} a+ib & c+id \\ -c+id & a-id\end{pmatrix}\right\}$$

(a_+, a_-) acts on $v \in \mathbb{H}$ by $a_- v a_+^{-1} \Rightarrow$ map to $SO(4)$.

$\phi_\pm \in \mathbb{H}_\pm$ by $a_\pm \phi_\pm$

just another copy of $\mathbb{H}$ with different rep.

Def $s \in \mathrm{Spin}^{(c)}(X)$ is a lift of an $SO(4)$ cocycle for TX to $\mathrm{Spin}^{(c)}(4)$.

Spinor bundles $W = W_+ \oplus W_-$ associated via reps $\mathbb{H}_+ \oplus \mathbb{H}_-$.

Determinant bundle $\det(s)$ associated to rep $\det(a_+)$ (i.e cx line bundle).

Clifford multiplication $\rho : TX \longrightarrow \mathrm{End}\, W$
at $x \in X$:

$$T_x X \oplus W_{+,x} \oplus W_{-,x}$$
$$\cong$$
$$\mathbb{H} \oplus \mathbb{H}_+ \oplus \mathbb{H}_-$$
$$(v, \phi_+, \phi_-) \longmapsto (v\phi_-, -\bar{v}\phi_+).$$

Has the property: $\rho(v)\rho(w) + \rho(w)\rho(v) = -2\langle v, w\rangle$.

<u>Note</u>
- spin^c case $\Rightarrow$ W a cx vector bundle where cx variable is I in $\mathbb{H}$. $\Rightarrow S^1$ action
- spin case $\Rightarrow$ W a quaternionic bundle. $\Rightarrow Sp(1) \supset S^1 \cup S^1 J = \mathrm{Pin}(2)$ action

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<u>Dirac operators</u>

<u>Def</u> A Clifford connection A on W is $\nabla^A(\rho(v)\phi) = \rho(v)\nabla^A\phi + \rho(\nabla^{LC} v)\phi$.

The space of Clifford conn is an affine space $\mathcal{A}(W)$ with model $i\Omega^1(X) = \Omega^1(X; i\mathbb{R})$

n.b this is Lie alg of S^1...

$(A_1 - A_0 = ia \otimes \mathrm{id}_W$ some $a \in \mathbb{R}.)$

Given a Clifford conn A

$$D_A : \Gamma(W) \xrightarrow{\nabla^A} \Gamma(T^*X \otimes W) \xrightarrow{\rho} \Gamma(W)$$

Note ρ switches $\pm \rightsquigarrow \mp$ $\Rightarrow$ so does D_A.

$A \in \mathcal{A}(W) \Rightarrow A^t$ conn on $\det(S)$

$\Rightarrow F_{A^t} \in \Omega^2(X; i\mathbb{R}) = i\Omega^2(X)$

$\Rightarrow F^+_{A^t} = \frac{1}{2}(F_{A^t} + *F_{A^t})$

Now given $\phi, \psi \in \Gamma(W)$ $\phi\phi^* : \Gamma(W) \to \Gamma(W)$; $\psi \mapsto \langle \phi, \psi \rangle \phi$

$\phi\phi^* \in \mathrm{End}(W^\pm)$ and has trace $|\phi|^2 \Rightarrow (\phi\phi^*)_0 = \phi\phi^* - \frac{1}{2}|\phi|^2 \mathrm{id}_{W^\pm}$

<u>FACT</u> ρ extends to an action of $\Lambda^* T^* X$ and identifies $i\Lambda^2_+ T^* M$ with trace-free skew-hermitian endos of $W^\pm$.

<u>Gauge group</u> $\mathcal{G} = C^\infty(X, S^1)$ acts on the <u>configuration space</u>

$$\mathcal{A}(W) \times \Gamma(W_+) \text{ by } u \cdot (A, \phi) = (A - u^{-1} du, u \cdot \phi).$$

and the equations are equivariant.

TRICK: don't work on the full configuration space, restrict to the <u>Coulomb slice</u>

$$\ker(d^*) \times \Gamma(W^+) \quad \text{fix a base conn } A_0 \in \mathcal{A}(W). \text{ Then } F_{A_0^t + a} = F_{A_0^t} + 2d^+ a$$

<u>FACT</u> For each (A, ϕ) $\exists!$ $u \in \{ e^{i\xi} \mid \xi \in C^\infty(X) \text{ and } \int_X \xi = 0 \}$ s.t. $(A, \phi) - u(A_0, \phi) \in \ker d^* \times \Gamma(W^+)$

Residual action of $S^1 \times H^1(X; \mathbb{Z})$.

[<u>FOR SIMPLICITY</u> We will always now have $b_1(X) = 0$]

<u>The monopole map</u>

$$\mu : \mathcal{C} = \ker d^* \times \Gamma(W^+) \longrightarrow i\Omega^2_+(X) \times \Gamma(W^-)$$
$$(a, \phi) \longmapsto (d^+ a + \rho^{-1}(\phi\phi^*)_0 + F^+_{A_0^t}, D^+_A \phi)$$

Ex: $D_A^+\phi = D_{A_0}^+\phi + \rho(a)\phi$

Some properties

$$\mu\begin{pmatrix} a \\ \phi \end{pmatrix} = \underbrace{\begin{pmatrix} d^+a \\ D_{A_0}^+\phi \end{pmatrix}}_{= L} + \underbrace{\begin{pmatrix} \rho^{-1}(\phi\phi^*)_0 + F_{A_0^t}^+ \\ \rho(a)\phi \end{pmatrix}}_{= Q}$$ "linear + quadratic"

- After suitable Sobolev completions L is linear Fredholm. $\Rightarrow$ take index WE MEAN $\mathbb{R}$-dim index
$\mathrm{ind}(L) = \mathrm{ind}_{\mathbb{R}}(L)$.

By Atiyah-Singer we get

$$\mathrm{ind}(L) = \tfrac{1}{4}(c_1^2(s) - \sigma(X)) + \underbrace{b_1(X)}_{=0 \text{ for us}} - b_2^+(X))$$

- Q is compact. i.e. Q(bounded) = relatively compact.
$\Rightarrow \mu^{-1}$(bounded) = bounded (using $b_1 = 0$)
$\Rightarrow$ can pass to bounded 1-point completions (a version of 1-pt cmpct for bounded but not locally cmpct spaces)

$$\mu^+ : \mathcal{C}^+ \to \mathcal{U}^+ = S_1(\mathcal{U} \oplus \mathbb{R})$$

$\Rightarrow \mu^+$ a map of (Hilbert) spheres. But in ∞ dim the spheres are contractible.

Lecture 2

Now set $G = S^1$ or $Pin(2) \equiv S^1 \cup S^1$
μ is G-equivariant w.r.t. scalar mult on spinors

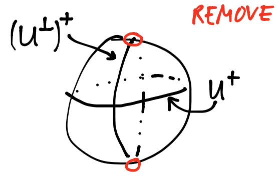
IDEA Restrict to fin dim subspaces to get stable htpy information.

Def $U \subset \mathcal{U}$ a fin dim subspace is admissible if
(a) U contains $L(\mathcal{C})^\perp$ a.k.a. $L \pitchfork U$.
(b) $\mu : L^{-1}(U) \to \mathcal{U}$ misses $S_1(U^\perp)$ unit sphere.

Why (a)? $\rightsquigarrow$ allows to $L^{-1}(U)$ relation U.
Why (b)? Need lemma:

Lemma For any $\overset{\text{fin dim}}{U} \subset \mathcal{U}$, $U^+ \subset \mathcal{U}^+ \setminus S_1(U^\perp)$ is G-equiv def retr

Proof idea $\mathcal{U} = \mathbb{R}^2$, $U = \mathbb{R}$



"$\square$"

Take a retraction $r_U : \mathcal{U}^+ \setminus S_1(U^\perp) \longrightarrow U^\perp$.

Then for $U \subset \mathcal{U}$ admissible:

$$\begin{tikzcd} L^{-1}(U)^+ \arrow[r, "\mu"] \arrow[dr, "\mu_U^+"'] & \mathcal{U}^+ \setminus S_1(U^+) \arrow[d] \\ & U^+ \end{tikzcd}$$

define μ_U^+

Lemma Any fin dim G-subspace of $\mathcal{U}$ is contained in an admissible one.

Proof $U_0 \subset \mathcal{U}$ arbitrary $\Rightarrow$ take $U_0 + (L(\mathcal{V})^\perp)$. $\therefore$ (a) = ✓

Take $\overset{\text{unit}}{\text{ball}}$ $B_1(\mathcal{U})$. Then $\mu^{-1}(B_1(\mathcal{U}))$ bounded $\Rightarrow$ $\overline{Q(\underbrace{\mu^{-1}(B_1(\mathcal{U}))}_{=:C}} \subset \mathcal{U}$ is compact.

$\Rightarrow$ can cover C by ε-n'hoods of fin many G-orbits, say orbits of c_i under G.

Claim: $U_0 + L(\mathcal{V})^\perp + G\text{-span}\{c_i\} =: U$ is admissible

Proof Take $w \in L^{-1}(U)$ and assume $\mu(w) \in S_1(U^\perp)$ $\Rightarrow$ $\mu(w) = L(w) + Q(w) \in U + C$

Now $\mu(w)$ is ε-close to U ⚡ for small ε. $\square$

$\square$

Consequently we get

$$\mu_U^+ : L^{-1}(U)^+ \longrightarrow U^+ \text{ for suff large } U.$$

Prop Suppose $U \subset W \subset \mathcal{U}$ admissible. $W = U \oplus V$ then the diagram

$$\begin{tikzcd} L^{-1}(W)^+ \arrow[r, "\mu_W^+"] \arrow[d, "\simeq"] & W^+ \arrow[d, "\simeq"] \\ V^+ \wedge L^{-1}(U)^+ \arrow[r, "\mathrm{id} \wedge \mu_U^+"'] & V^+ \wedge U^+ \end{tikzcd}$$

commutes up to homotopy

Remark The G-equivariant version of suspension. "Susp by V"

In other words: For admissible $\mathcal{U}$, we get

$$[\mu] \in \{L^{-1}(\mathcal{U})^+, \mathcal{U}^+\}^G_{\mathcal{U}} = \underset{\substack{V \subset \mathcal{U}^\perp \\ \text{fin dim}}}{\mathrm{colim}} [V^+ \wedge L^{-1}(\mathcal{U})^+, V^+ \wedge \mathcal{U}^+]^G_0$$ i.e. pointed

How do we use this? Let's see what $\mathcal{U}$ and $L^{-1}(\mathcal{U})$ look like.

<u>spin^c case</u> • any fin dim subspace of $\mathcal{U}$ or $\mathcal{C}$ is of the form $\underset{\substack{\text{triv} \\ S^1\text{rep}}}{\mathbb{R}^m} \oplus \underset{\substack{\text{stnd rep} \\ \text{on } \mathbb{C}}}{\mathbb{C}^n}$

• $\mathcal{U} \cong \mathbb{R}^{m+b_2^+} \oplus \mathbb{C}^n$ admissible $\Longrightarrow$

$$\mathcal{U} \cong \mathcal{U}' \oplus \mathrm{coker}(d^+) \oplus \mathrm{coker}(D^+_{A_0})$$
$$L^{-1}(\mathcal{U}) \cong \mathcal{U}' \oplus \ker(d^+) \oplus \ker(D^+_{A_0}).$$

Hodge theory $\Rightarrow$ $\mathrm{coker}(d^+)$ has dim b_2^+, $\dim \ker d^+ = b_1 = 0$

$$\mathrm{ind}_{\mathbb{C}}(D_A) = \frac{c_1^2(s) - \sigma(X)}{8} =: d$$
$$\Rightarrow L^{-1}(\mathcal{U}) \cong \mathbb{R}^m \oplus \mathbb{C}^{n+d}$$
$$\Rightarrow [\mu] \in \{(\mathbb{R}^m \oplus \mathbb{C}^{n+d})^+, (\mathbb{R}^{n+b_2^+} \oplus \mathbb{C}^n)^+\}^{S^1}_{\mathcal{U}}$$

So if $b_2^+ = 0$, S^1-equivariant topology tells us that there is no such map unless $d \leq 0$.
(uses S^1-equiv K-theory)

But
$$d \leq 0 \iff c_1^2(s) - \sigma(X) \leq 0.$$
$\Rightarrow Q_X$ is diagonalisable (We recover Donaldson)

When $b_2^+ \neq 0$ we get more generally SW-invariants.

$$[\mu] \in \{\mathbb{C}^d, \mathbb{R}^{b_2^+}\}^{S^1}_{\mathcal{U}} = \pi^{b_2^+}_{S^1,\mathcal{U}}(\mathbb{C}^d) \overset{b_2^+>0}{\cong} \pi^{b_2^+-1}_{st}(\mathbb{C}P^{d-1})$$
$$\downarrow \text{Hurewicz}$$
$$H^{b_2^+-1}(\mathbb{C}P^{d-1}, \mathbb{Z})$$
$$SW(s) \in \overset{\cong}{\mathbb{Z}}$$

Lecture 3

References

- Bauer-Furuta: stable cohomotopy refinement
- Bauer : Refined SW
- Furuta : $11/8$

$\}$ 4d stuff

- Manolescu : SW Floer stable htpy type, triangulation conjecture
- Khandhawit : ... new gauge slice ... also his thesis

$\}$ 3d stuff

Recall the philosophy is that 4-mflds $\rightsquigarrow$ stable htpy classes
3-mflds $\rightsquigarrow$ stable htpy types

Today Y is a closed oriented 3-mfld with $b_1(Y)=0$.

Study 4-dim SW eq'ns for $\hat{Y}=\mathbb{R}\times Y$

In 3d $\mathrm{Spin}^c(3)=U(2)\supset SU(2)=\mathrm{Spin}(3)$
$\Rightarrow$ only one spinor bundle W (trivial bundle).

In particular, can fix a flat base connection A_0 on W.

The SW equations for $(\hat{a},\hat{\phi})\in i\Omega^1(\hat{Y})\times\Gamma(\hat{W}^+)$ (identifying W with $\hat{W}^+$ somehow...)

So we instead look at 1-param family $(a_t,\phi_t)_{t\in\mathbb{R}}\in i\Omega^1(Y)\times\Gamma(W)$

and interpreted as such, we see SW as flow equations:

$$(a,\phi)\mapsto(*da+\rho^{-1}(\phi\phi^*)_0,\ D_{A_0}\phi+\rho(a)\phi).$$

We now have to play the same game as for 4dim.

- Coulomb slice $\mathcal{U}:=\ker d^*\times\Gamma(W)$

$$\Rightarrow \mathcal{F}:=\underbrace{(*da,D_{A_0}\phi)}_{=:L}+\underbrace{(\rho^{-1}(\phi\phi^*)_0-d\xi(\phi)\ ,\ \rho(a)\phi-\xi(\phi)\phi)}_{=:Q}$$

(now can be sort of cubic...)

where $\xi(\phi)\in C^\infty(Y)$ s.t. $\Delta\xi(\phi)=d^*\rho^{-1}(\phi\phi^*)_0$, $\int_Y\xi(\phi)=0$.

$\Rightarrow \mathcal{F} : \mathcal{U} \to \mathcal{U}$ equivariant wrt residual gauge action by $\underset{\text{Pin}_2}{\overset{S^1}{\text{or}}} \times \underbrace{H^1(Y;\mathbb{Z})}_{\text{assume } = 0.}$

- Before, we had L Fredholm and Q nice compactness properties.

  Now (using $b_1(Y) = 0$)

  $\mathcal{F} = \text{grad}\,\mathcal{L}$, $\mathcal{L} : \mathcal{U} \to \mathbb{R}$ (cf. Chern–Simons–Dirac functional)

  $S(\mathcal{F}) = \left\{\begin{matrix}\text{crit points} \\ + \text{trajectories}\end{matrix}\right\}$ is bounded

  $\cap$

  $\mathcal{R} := B_R(\mathcal{U})$, $R >> 0$ an R-ball. (note are not in a discrete critical point pertubation setting.)

- Finite dim approx

  $U \subset \mathcal{U}$ fin dim G-inv, $\pi_U : \mathcal{U} \to U$ projection

  $\Rightarrow \mathcal{F}_U : \pi_U \circ \mathcal{F}_U : U \to U$ generates a flow on U.

<u>FACTS</u> • For large U, $S_U := S(\mathcal{F}_U) \subset \mathcal{R} \cap U$ (and this guy is compact, where of course U was not)

- Define $E_U = (U_+)^+ \wedge \underbrace{CI(\mathcal{F}_U, S_U)}_{\text{The Conley Index. It is a pointed } G\text{-space.}}$

What is $\underset{??\uparrow?}{U_+}$. $\mathcal{U} = \mathcal{U}_+ \oplus \mathcal{U}_-$ wrt the self-adjoint operator L (possibly?)

If $W = U \oplus V$ then

$$
\begin{aligned}
E_W &= (W_+)^+ \wedge CI(\mathcal{F}_W, S_W) \\
&= (W_+)^+ \wedge (V_-)^+ \wedge CI(\mathcal{F}_U, S_U) \\
&= (V_+)^+ \wedge (V_-)^+ \wedge (U_+)^+ \wedge CI(\mathcal{F}_U, S_U) \\
&= V^+ \wedge E_U
\end{aligned}
$$

<u>In other words</u> $U \mapsto E_U$ defines a G-equivariant spectrum parametrised by the "universe" $\mathcal{U}$.

Conley index theory (there's also an equivariant version, natürlich.)

Input: ① flow on a locally compact metric space

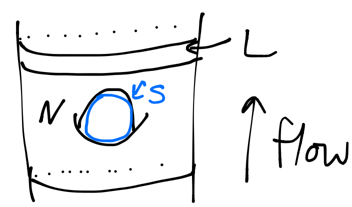
$$\varphi : \Omega \times \mathbb{R} \to \Omega$$

② $S \subset \Omega$ compact, isolated flow-invariant set

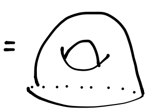
Def Isolated means can find small n'hood in which it is the biggest flow-invariant set

Output: Pointed htpy type $CI(\varphi, S) = [N/L]$ where (N, L) is an index pair for S.

Def An index pair
- $S \subset N \setminus L$
- L +tively invariant w.r.t N.
- L is an "exit set" for N



Thm (Conley) $CI(\varphi, S)$ is independent of the index pair.

$\Rightarrow N/L =$  $\simeq S^1 \vee S^1$

CONLEY THEORY IS BYPASSING TRANSVERSALITY

– in particular we aren't perturbing out of Morse-Bott (or whatever) we don't care about anything like this.

So far we get an invariant of (Y, s, g).

Varying $g \Rightarrow$ we need to suspend by $\mathbb{C}^{st}$.

To resolve this we pass to the "Manolescu category":

objects: (A, m, n) "=" $\Sigma^{-\mathbb{R}^m} \Sigma^{\mathbb{C}^n} A$

A: ptd space, $(m, n) \in \mathbb{Z} \times \mathbb{Q}$

$\Rightarrow SWF(Y, s) = (E_u, 0, n(Y, s, g))$

Thm (Lidman-Manolescu) $\tilde{H}^{S^1}_*(SWF(Y,s)) \cong \widehat{HM}_*(Y, s) \cong HF^-_*(Y, s)$

Where recall there is action on LHS by $H^*_{S^1}(pt) = H^*(\mathbb{CP}^\infty) \cong \mathbb{Z}[u]$.

<u>Remark</u> Get d-invariants in terms of localising LHS at u.
